# Steepest descent applied to the linearised KdV equation.

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### I - KdV equation - Dispersive waves.

The Kortewegde Vries equation, also known as KdV, is given by

$$u_t + u_{xxx} + 6uu_x = 0,$$

and describes the behavior of waves in shallow water. We will find some solutions for the linearized KdV equation, that is

$$u_t + u_{xxx} = 0,$$

with the initial data $u(x,0) = f(x)$, and $f$ analytic. Using Fourier transform on KdV we find

$$\hat{u}_t + (ik)^3 \hat{u} = 0 \quad \text{with} \quad \hat{u}(k,0) = \hat{f}(k).$$

Then

$$\hat{u}(k,t) = \hat{f}(k) e^{-(ik)^3 t}, \quad \text{and} \quad u(x,t) = \frac{1}{2\pi} \int_0^\infty \hat{f}(k) e^{i(k^3 t + ikx)} \, dk.$$

Now, let

$$\phi(k) = i\left(k^3 + ik\frac{x}{t}\right),$$

so that we have the solution in an integral form appropriate for the application of the method of steepest descent:

$$u(x,t) = \frac{1}{2\pi} \int_0^\infty \hat{f}(k) e^{t\phi(k)} \, dk.$$

It is now possible to find the behaviour of $u(x,t)$ as $t \to \infty$ with $x/t$ fixed.

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### II-Application of the method.

There are two possible situations: $(x/t) < 0$ and $(x/t) > 0$ the procedure is the same. We will work on the case $(x/t) < 0$. In this case the saddle points are

$$k_L = -\left|\frac{x}{3t}\right|^{1/2}, \quad \text{and} \quad k_R = \left|\frac{x}{3t}\right|^{1/2}.$$

Now we deform the real line into the union of two contours that follow the paths of steepest descent. One contour goes through $k_L$, starts on the lowest point of the upper left half plane and ends just to the left of the imaginary axis on the lower half plane. The other contour, which goes through $k_R$, is the mirror image of the first one respect of the imaginary axis. The two contours have contributions of the same order $u_L(x,t)$ and $u_R(x,t)$ to $u$, thus

$$u(x,t) = u_L(x,t) + u_R(x,t).$$

For each saddle we have $\phi''(k_L) = -i6|x/(3t)|^{1/2}$, and $\phi''(k_R) = i6|x/(3t)|^{1/2}$.

Using the standard expression for the contribution of a saddle we find for $k_L$ with $x/t$ fixed

$$u_L(x,t) \sim \frac{\hat{f}(-|\frac{x}{3t}|^{1/2})e^{2it|\frac{x}{3t}|^{3/2}-i\frac{\pi}{4}}}{\sqrt{12\pi t}\ |\frac{x}{3t}|^{1/4}}\left(1+\mathcal{O}(t^{-1})\right) \quad \text{as} \quad t \to \infty,$$

and similarly for $k_R$

$$u_R(x,t) \sim \frac{\hat{f}(|\frac{x}{3t}|^{1/2})e^{-2it|\frac{x}{3t}|^{3/2}+i\frac{\pi}{4}}}{\sqrt{12\pi t}\ |\frac{x}{3t}|^{1/4}}\left(1+\mathcal{O}(t^{-1})\right) \quad \text{as} \quad t \to \infty.$$

Noting that the conjugate of the Fourier transform obeys $\hat{f}(-k) = \hat{f}(k)^{\dagger}$, and defining it $\hat{f}(|\frac{x}{3t}|^{1/2}) = r(|\frac{x}{3t}|^{1/2})e^{i\theta_0}$ is possible to write the solution $u(x,t)$ as

$$u(x,t) = u_L(x,t)+u_R(x,t) \sim \frac{2r(|\frac{x}{3t}|^{1/2})}{\sqrt{12\pi t}\ |\frac{x}{3t}|^{1/4}}\cos\left(2t\left|\frac{x}{3t}\right|^{3/2} - \frac{\pi}{4} - \theta_0\right)\left(1+\mathcal{O}(t^{-1})\right) \quad \text{as} \quad t \to \infty.$$

The case $x/t > 0$ yields exponential behaviour.

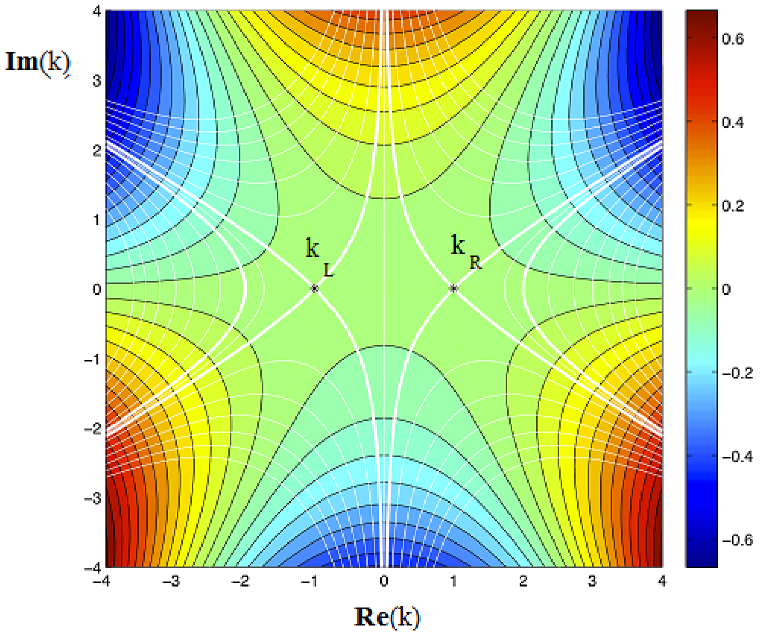


Figure 1: Steepest descent contours of the solutions of the KdV equation for $x/t < 0$ . (Courtesy Prof. David J. Muraki–Courant Institute NYU and Simon Fraser University)