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Twisted Elastic Rings and the Rediscoveries of Michell's Instability

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Abstract Elastic rings become unstable when sufficiently twisted. This fundamental instability plays an important role in the modeling of DNA mechanics and in cable engineering. In 1962, Zajac computed the value of the critical twist for the instability. This critical value was rediscovered in 1979 by Benham and independently by Le Bret in elastic models for DNA; unstable rings have since become an important example of elastic instabilities in rods both for the development of new methods and in applications. The purpose of this note is to show that the problem had been completely solved by John Henry Michell in 1889 in a rather elegant manner and to reflect on its history and modern developments.

Key words: Kirchhoff rods · elastic rings · twist · instability

1. Introduction

The problem is simple to state: "If a wire of isotropic section and naturally straight be twisted, and the ends joined so as to form a continuous curve, the circle will be a stable form of equilibrium for less than a certain amount of twist." [1] In other words, consider an isotropic elastic rod (the rod has no preferred bending direction[1]) that is stress-free when held straight. Now, paint a straight line on the straight rod and shape the rod so that the rod centerline is a circle. At the junction, the tangent from the two ends agree but the cross-sections can be rotated so that the line painted on the straight unstressed shape twists around the central curve. The twist is the total angular rotation of the line with respect to the central curve. The line will close on

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[1] See [2] for a modern mathematical definition of transverse isotropy in elastic rods.

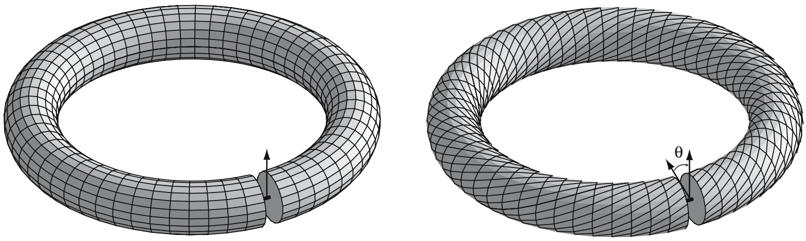

Figure 1 Untwisted (*left*) and twisted ring (*right*). The total twist is given by the total angular rotation of an *arrow pointing* at any *circle* on the surface of the untwisted ring. Here the total twist is $\frac{13}{2}\pi$.

itself at the junction if the twist is an integral multiple of 2π (Figure 1). The rod is glued at this point and released. For small values of the twist, the twisted ring is stable. For sufficiently high twist, the elastic ring will become unstable and will start writhing out of the plane. The phenomenon is quite striking as the instability appears to be subcritical (in the sense that no stable equilibrium shape exists close to the unstable ring). The ring suddenly buckles and loops back on itself by forming an eight-shape where self-contact plays a particularly important role (see Figure 2). For reasons that will soon become apparent, we shall refer to the twisted elastic ring instability as *Michell's instability* and the problem addressed here is to identify the value of the critical twist at which the instability sets in.

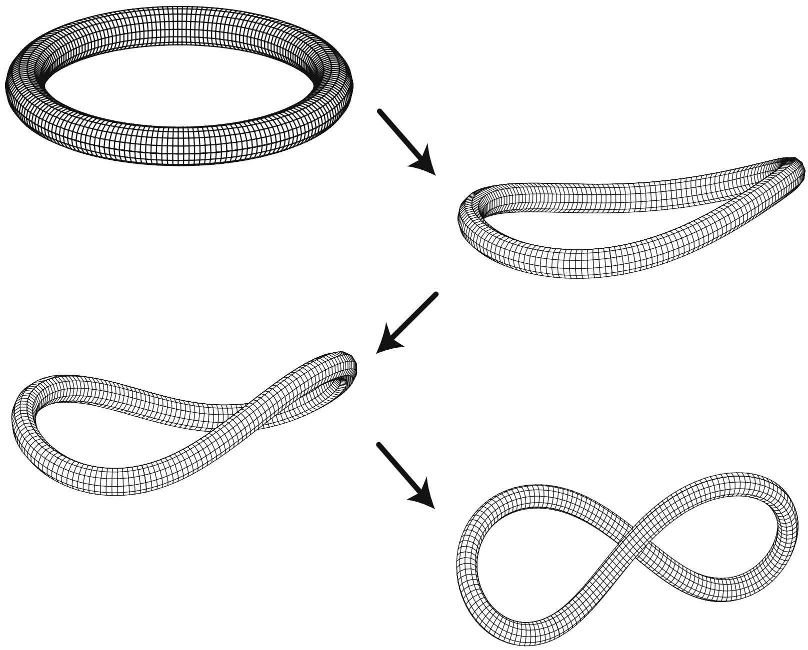

Figure 2 Buckling of an unstable ring when the critical twist is first reached (twist not shown on the ring).

This phenomenon raises many interesting questions. Qualitatively, one may understand the instability as a balance between torsional and bending energy. The torsional energy of the ring increases as the square of the twist and is eventually relieved by a change of shape that increases the bending energy (proportional to the square of the curvature). Quantitatively, the first natural question is to determine the value of the critical twist that makes the ring unstable in terms of its geometric (ring and cross-section radii) and elastic parameters (torsional and flexural rigidity). Once the instability threshold has been determined, many other questions can be approached such as determining the static and dynamic behavior of the ring after the instability sets in and how this phenomenon can be generalized to more complicated systems (see below).

The instability of the twisted elastic ring is a fundamental instability of elastic materials akin to the Euler instability describing the buckling of loaded beams. Beyond its obvious importance as a natural philosophy question and its application in engineering problems, the problem of twisted elastic rings and related instability of elastic materials has gained some renewed interest in science largely due to the realization that Kirchhoff models for elastic rods are suitable models for the study of macromolecules such as DNA molecules [3–6] but also plants [7, 8] and microbial filaments [9]. In particular, the analysis of mini-DNA rings made out of a few hundred bases offers a unique perspective to characterize physical properties of DNA and twisted elastic rings are the natural theory to understand and extract these properties [10].

2. History of the Problem

The stability of twisted rings was first discussed by Thomson and Tait in their classical *Treatise on Natural Philosophy* [11]. In paragraph 123 (see Figure 3), they discussed the problem of the respective stability of the circle *versus* the eight form (close to the last shape shown on Figure 2) and reached the conclusion that "the circular form, which is always a figure of equilibrium, may be stable or unstable, according as the ratio of torsional to flexural rigidity is more or less than a certain value depending on the actual degree of twist." Motivated by this assertion, John Henry Michell (Figure 4) wrote a four-page paper where he determined the critical twist as a function of the ratio $\alpha = M/L$ of torsional to flexural rigidity. His original paper is given in Appendix A and a modern proof based on his analysis is given in the next Section. His analysis rests on an application of a general theory of vibration of rods around an equilibrium shape [12] (his first published work at age 26). Apparently, Michell realized that when these frequencies become imaginary the equilibrium shape loses its stability and he applied this idea to derive a simple criterion for the instability of a twisted elastic ring.

John Henry Michell is an interesting, almost tragic, figure of applied mathematics at the turn of the 20th century. A bright Australian student, he went to Cambridge (UK) for his postgraduate study and then returned to the University of Melbourne where he was eventually appointed Professor of Mathematics and retired at age 65 [13]. His entire research publication records took place between 1889 and 1902 when he published 23 papers. His contributions are believed to be "the most important contributions ever made by an Australian mathematician" [14]. While Michell was very active in teaching and science in Australia after 1902, the reasons of his abandonment of research activity are unclear and may be due to his dedication to

The principles of twist thus developed are of vital importance in the theory of rope-making, especially the construction and the dynamics of wire ropes and submarine cables, elastic bars, and spiral springs.

For example: take a piece of steel pianoforte-wire carefully straightened, so that when free from stress it is straight: bend it into a circle and join the ends securely so that there can be no turning of one relatively to the other. Do this first without torsion: then twist the ring into a figure of 8, and tie the two parts together at the crossing. The area of the spherical hodograph is zero, and therefore there is one full turn (2π) of twist; which (§ 600 below) is uniformly distributed throughout the length of the wire. The form of the wire, (which is not in a plane,) will be investigated in § 610. Meantime we can see that the "torsional couples" in the normal sections farthest from the crossing give rise to forces by which the tie at the crossing is pulled in opposite directions perpendicular to the plane of the crossing. Thus if the tie is cut the wire springs back into the circular form. Now do the same thing again, beginning with a straight wire, but giving it one full turn (2π) of twist before bending it into the circle. The wire will stay in the 8 form without any pull on the tie. Whether the circular or the 8 form is stable or unstable depends on the relations between torsional and flexural rigidity. If the torsional rigidity is small in comparison with the flexural rigidity [as (§§ 703, 704, 705, 709) would be the case if, instead of round wire, a rod of + shaped section were used], the circular form would be stable, the 8 unstable.

Lastly, suppose any degree of twist, either more or less than 2π, to be given before bending into the circle. The circular form, which is always a figure of free equilibrium, may be stable or unstable, according as the ratio of torsional to flexural rigidity is more or less than a certain value depending on the actual degree of twist. The tortuous 8 form is not (except in the case of whole twist $= 2\pi$, when it becomes the plane elastic lemniscate of Fig. 4, § 610,) a continuous figure of free equilibrium, but involves a positive pressure of the two crossing parts on one another when the twist $> 2\pi$, and a negative pressure (or a pull on the tie) between them when twist $< 2\pi$: and with this force it is a figure of stable equilibrium.

Figure 3 Extract from Trait and Thomson's Natural Philosophy (1867).

teaching and a lack of positive response from the scientific community [15, 16]. To date, his single most recognized work is the computation of wave resistance to a ship, that is, the energy loss into a wave pattern by a steadily moving ship known as the "wave resistance formula," which was not fully appreciated before his death [14].

Michell's work on the stability of rods received some attention at the turn of the 20th century and eventually led him, with other contributions, to his election at the Royal Society in 1902. At this time his work in elasticity was well received as is evidenced by the discussion in Basset's paper [17] and in the second edition (1906)

Figure 4. John Henry Michell (1863–1940).

of Love's classic treatise [18]. However, his pioneering work in elasticity seems to have fallen in complete darkness as it is completely absent from the literature after 1945 (with the notable exception of Antman and Kenney [19]). At the turn of the 21st century, the name of John Henry Michell has resurfaced in Australia through the establishment of the J.H. Michell Medal, awarded yearly since 1999 by the Australian Mathematical Society to a young outstanding applied mathematician.

In 1962, Edward E. Zajac, then at Bell Laboratory, published an article on the stability of twisted elastic rings and the stability of the clamped looped elastica [20]. His work was motivated by the coiling and kinking of submarine cables but he argued that these problems 'are of intrinsic interest in applied mechanics.' He was apparently unaware of Michell's work as he states that "contrary to what one might expect, neither of the foregoing problems is solved in Born's thesis on the stability of elastic line." In the first part of the paper, Zajac studies the stability of the twisted elastic ring based on Love's formulation in Euler angles and rederives Michell's criterion by linearizing the static equation of rod equilibrium. His paper is clear and concise and a good example of applied mechanics at his best, a well-formulated problem of interest solved elegantly by direct analysis of Kirchhoff equations.

The story of repeated discoveries does not end with Zajac. Zajac's paper was published in an engineering journal and aside from the submarine cable community it did not receive much attention. However, the stability of twisted elastic rings became of interest to the biophysics DNA community when it was first realized that geometric and topological characterizations of curves could be of importance to understand DNA configurations [21, 22]. Shortly after, Benham and LeBret independently proposed to model DNA as an elastic rod [3, 23, 24] and both considered the stability of twisted elastic rings and essentially rederived Michell's criterion [25, 26]. The connection with Zajac's work was only realized years later by Coleman, Tobias, Olson, and collaborators in a series of papers [27–29]. Since then, Zajac's work has been considered as the original paper on the subject [30–35] and the instability of the twisted rings has even been referred to as Zajac's instability [36].

3. Michell's Analysis

The basic idea behind Michell's analysis is to study the linearized dynamics of the rings and to identify vibration frequencies. The instability threshold is reached when the frequencies become imaginary, that is when small perturbations are exponentially amplified. Therefore, the analysis should be based on the full dynamical equations (described in the next section). However, in Michell's analysis, the rotational acceleration of the cross-section, given by the right-hand side of Equation (16), is not taken into account. This leads to a simpler formulation (independent of the spin vector **w**) that gives the wrong vibration frequencies, but since the stability threshold does not depend on the dynamical part of the equations but only on the static part, it does not change the basic computation for the critical value of the twist. Accordingly, to follow closely Michell's basic analysis while remaining mathematically consistent, we give a simpler and shorter proof by considering the static equations and looking for non-trivial periodic solutions of the linearized equation. This proof is actually very close to the derivation of Euler criterion for the instability of a beam and to the best of my knowledge the simplest self-contained proof available.

In the absence of body force, the static equations describing the balance of resultant force[2] $\mathbf{n}$ and moment $\mathbf{m}$ acting on the centerline $\mathbf{r} = \mathbf{r}(s)$ parameterized by its arc length are given by

$$\mathbf{n}' = 0, \tag{1}$$

$$\mathbf{m}' + \mathbf{r}' \times \mathbf{n} = 0. \tag{2}$$

In the case of an isotropic cross-section, these equations are closed by the constitutive relationship

$$\mathbf{m} = L\kappa\boldsymbol{\beta} + M\gamma\boldsymbol{\tau} \tag{3}$$

where $\mathbf{r}' = \boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ are, respectively, the tangent, normal, and binormal vectors to $\mathbf{r}$. We define τ, κ to be the Frenet torsion and curvature and γ is the twist. They are related by the Frenet equations:

$$\boldsymbol{\tau}' = \kappa\boldsymbol{\nu}, \quad \boldsymbol{\nu}' = \tau\boldsymbol{\beta} - \kappa\boldsymbol{\tau}, \quad \boldsymbol{\beta}' = -\tau\boldsymbol{\nu}. \tag{4}$$

The twist γ describes the rotation of the material frame around the centerline and therefore is not a property of the centerline. Therefore, it does not appear in the Frenet equations but only in the constitutive relationship. The coefficients M and L describe the bending and torsional rigidity of the rod. By substituting the constitutive equation into Equations (1–2) and using the Frenet equation, one can solve for $\mathbf{n}$ explicitly in terms of κ, τ, γ and reduce the system to three differential equations in terms of geometric parameters:

$$\gamma' = 0 \tag{5}$$

$$\kappa\tau' - \alpha\gamma\kappa' = 0 \tag{6}$$

$$\kappa^2(\alpha\gamma - 2\tau)\tau' + \kappa'(\kappa^3 - \kappa'') + \kappa\kappa''' = 0 \tag{7}$$

where $\alpha = M/L$. The first equation implies that the twist γ must be constant. To identify bifurcations points where the ring may lose stability with respect to out-of-plane and torsional deformations we consider small variations of the curvature and

[2] Michell uses S, T, U and F, G, H to denote the components of $\mathbf{n}$ and $\mathbf{m}$ in the Frenet basis, see Appendix A.

torsion $\kappa = K + \epsilon\kappa_1, \tau = \epsilon\tau_1$ and expand Equations (6–7) to first order in ϵ (where $K = 1/R$ is the curvature of the circle). After simplification, a single equation for κ_1 is obtained

$$\kappa_1''' + \left(K^2 + \alpha^2\gamma^2\right)\kappa_1' = 0. \tag{8}$$

This equation supports periodic solutions of the form $\kappa_1 = \exp(insK)$ with n integer for

$$\alpha^2\gamma^2 K^{-2} = n^2 - 1. \tag{9}$$

The first non-trivial solution occurs for $n = 2$ and corresponds to

$$\gamma = \sqrt{3}\frac{K}{\alpha}. \tag{10}$$

The total twist Tw is the integral of the twist γ over the circumference and corresponds to the total angular rotation of the cross-section along the rod, that is $Tw = \frac{2\pi\gamma}{K}$ and the twisted ring becomes unstable for values of the twist larger than

$$Tw_c = \frac{2\pi\sqrt{3}}{\alpha}. \tag{11}$$

This is the critical value of the twist identified by J. H. Michell in 1889.

For most materials, typical values for α lie between 2/3 (incompressible material) and 1 (compressible) with metals around 4/5. Other filaments such as DNA may present higher values of twist to bending rigidity [31]. Therefore, a ring becomes unstable when it has been twisted by about two full turns (with limits 1.73 to 2.6 corresponding to α between 2/3 and 1).

The stability of the ring solution with respect to the out-of-plane deformation cannot be assessed by the present method. This can be achieved either by looking at the dynamics of the perturbed solutions and verifying that past the critical values, these solutions are exponentially growing in time. This was the method originally proposed by Michell and has been developed independently in a general framework by Goriely and Tabor [33, 37–39]. The second alternative to test for stability is to use the variational structure of the Kirchhoff equations and identify the mimina of the corresponding energy functional. The powerful machinery of variational calculus presents some interesting subtleties in the case of rods due to the particular integral constraints associated with inextensibility and unshearability [10, 40–43]. For a geometric variational proof of stability, see [44].

4. A Modern Formulation of Rod Theory

Here, following [2, 10, 27], we give a short overview of the general theory of rods in modern notation. A *Cosserat* or *Kirchhoff rod* is represented by its centerline $\mathbf{r}(s)$ where s is a material parameter taken to be the arc length in a stress-free configuration $(0 \leq s \leq L)$ and two orthonormal vector fields $\mathbf{d}_1(s), \mathbf{d}_2(s)$ representing the orientation of a material cross-section at s. A local orthonormal basis is obtained by defining $\mathbf{r}' = \mathbf{d}_3(s) = \mathbf{d}_1(s) \times \mathbf{d}_2(s)$ and a complete kinetic and dynamics description is given by

$$\mathbf{r}' = \mathbf{v}, \tag{12}$$

$$\mathbf{d}_i' = \mathbf{u} \times \mathbf{d}_i, \quad i = 1,2,3, \tag{13}$$

$$\dot{\mathbf{d}}_i = \mathbf{w} \times \mathbf{d}_i \quad i = 1,2,3, \tag{14}$$

where $()'$ and $(\dot{\ })$ denote the derivative with respect to s and t, and $\mathbf{u}, \mathbf{v}$ are the *strain* vectors and $\mathbf{w}$ is the *spin* vector. The components of a vector $\mathbf{a} = \mathsf{a}_1\mathbf{d}_1 + \mathsf{a}_2\mathbf{d}_2 + \mathsf{a}_3\mathbf{d}_3$ in the local basis are denoted by $\mathsf{a} = (\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3)$ (following [2], we use the *sans-serif fonts* to denote the components of a vector in the local basis). The two first components represent transverse shearing while $\mathsf{v}_3 > 0$ is associated with stretching and compression. The two first components of the *curvature vector* u, are associated with bending while u_3 represents twisting.

The stress acting at $\mathbf{r}(s)$ is given by a resultant force $\mathbf{n}(s)$ and resultant moment $\mathbf{m}(s)$. The balance of linear and angular momenta yields [2]

$$\mathbf{n}' + \mathbf{f} = \rho \mathcal{A} \ddot{\mathbf{r}}, \tag{15}$$

$$\mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \mathbf{l} = \rho\Big(I_2\mathbf{d}_1 \times \ddot{\mathbf{d}}_1 + I_1\mathbf{d}_2 \times \ddot{\mathbf{d}}_2\Big), \tag{16}$$

where $\mathbf{f}(s)$ and $\mathbf{l}(s)$ are the body force and couple per unit length applied on the cross-section at s, $\mathcal{A}$ is the cross-section surface, ρ the mass density, and $I_{1,2}$ are the principal moments of inertia of the cross-section (corresponding to the directions $\mathbf{d}_{1,2}$).

To close the system, we assume that the resultant stresses are related to the strains. There are two important cases to distinguish.

4.1. Extensible and Shearable Rods

First, we consider the case where the rod is extensible and shearable and we assume that there exists a strain-energy density function $W = W(\mathsf{y}, \mathsf{z}, s)$ such that the constitutive relations for the resultant moment and force in the local basis are given by

$$\mathsf{m} = f(\mathsf{u} - \hat{\mathsf{u}}, \mathsf{v} - \hat{\mathsf{v}}, s) = \partial_{\mathsf{y}} W(\mathsf{u} - \hat{\mathsf{u}}, \mathsf{v} - \hat{\mathsf{v}}, s), \tag{17}$$

$$\mathsf{n} = g(\mathsf{u} - \hat{\mathsf{u}}, \mathsf{v} - \hat{\mathsf{v}}, s) = \partial_{\mathsf{z}} W(\mathsf{u} - \hat{\mathsf{u}}, \mathsf{v} - \hat{\mathsf{v}}, s), \tag{18}$$

where $\hat{\mathsf{v}}$, $\hat{\mathsf{u}}$ are the strains in the unstressed reference configuration ($\mathsf{m} = \mathsf{n} = 0$ when $\mathsf{u} = \hat{\mathsf{u}}, \mathsf{v} = \hat{\mathsf{v}}$). Typically, W is assumed to be continuously differentiable, convex, and coercive. The rod is *uniform* if its material properties do not change along its length (*i.e.*, W has no explicit dependence on s) and the stress-free strains $\hat{\mathsf{v}}, \hat{\mathsf{u}}$ are independent of s.

4.2. Inextensible and Unshearable Rods

In the second case, we assume that the rod is inextensible and unshearable (the case considered by Michell), that is we take $\mathbf{v} = \mathbf{d}_3$ and the material parameter s becomes the arc length. In that case, there is no constitutive relationship for the resultant force and the strain-energy density function is a function only of $(\mathsf{u}-\hat{\mathsf{u}})$, that is

$$\mathsf{m} = \partial_{\mathsf{y}} W(\mathsf{u}-\hat{\mathsf{u}}) = f(\mathsf{u}-\hat{\mathsf{u}}). \tag{19}$$

In Michell's case, the energy is quadratic, the stress-free configuration is the straight rod, and the cross-section is circular. Explicitly, the resultant moment is

$$\mathbf{m} = EI_1\mathsf{u}_1\mathbf{d}_1 + EI_2\mathsf{u}_2\mathbf{d}_2 + \mu J\mathsf{u}_3\mathbf{d}_3, \tag{20}$$

where E is the young modulus, μ is the shear modulus, and J is a parameter that depends on the cross-section shape (an explicit form for J and examples can be found in [50]). For a circular cross-section, these parameters are:

$$I_1 = I_2 = \frac{J}{2} = \frac{\pi R^4}{4}, \tag{21}$$

where R is the radius of the cross-section. The products EI_1 and EI_2 are usually called the *principal bending stiffnesses* of the rod, and μJ is the *torsional stiffness*.

The orthonormal frame $(\mathbf{d_1}, \mathbf{d_2}, \mathbf{d_3})$ is different from the Frenet–Serret frame defined by the triple (normal,binormal,tangent)=$(\boldsymbol{\nu}, \boldsymbol{\beta}, \boldsymbol{\tau})$. If we take $\mathsf{v}_3 = 1$, then the vectors $(\mathbf{d_1}, \mathbf{d_2})$ lie in the normal plane to the axis and are related to the normal and binormal vectors by a rotation through an angle φ:

$$\mathbf{d}_1 = \boldsymbol{\nu} \cos\varphi + \boldsymbol{\beta} \sin\varphi, \tag{22}$$

$$\mathbf{d}_2 = -\boldsymbol{\nu} \sin\varphi + \boldsymbol{\beta} \cos\varphi. \tag{23}$$

This rotation implies that

$$\mathbf{u} = (\kappa \sin\varphi, \kappa \cos\varphi, \tau + \varphi'), \tag{24}$$

where κ and τ are the usual Frenet curvature and torsion.

4.3. Michell's Analysis for Rods with Intrinsic Curvature

Michell's analysis can be generalized to consider the stability of various configurations. As an example, we consider here the stability of an inextensible rings with intrinsic curvature. Such rods are characterized by the constitutive relationship

$$\mathsf{m} = B_1(\mathsf{u}_1 - \hat{\mathsf{u}}_1)\mathbf{d}_1 + B_2(\mathsf{u}_2 - \hat{\mathsf{u}}_2)\mathbf{d}_2 + C\mathsf{u}_3\mathbf{d}_3, \tag{25}$$

where $\hat{\mathsf{u}}_{1,2}$ are constant. The stability analysis can be carried out in the general case but we only present here a simpler problem to avoid lengthy discussions on parameters. We consider an isotropic rod $(B_1 = B_2 = B)$ and choose the director basis so that the first vector is in the direction of intrinsic curvature, that is $\hat{\mathsf{u}}_2 = 0$. In this case, there exists a family of rings with arbitrary curvature K defined by $\mathsf{u}_1 = K$,

$u_2 = u_3 = 0$. Following Michell's idea, we can solve for the resultant force and write a closed system for the curvatures. Explicitly, we first substitute, the constitutive relationship (25) into the equation for the resultant moment written in local components. The equation for the tangential component m_3 leads to

$$\alpha u_3' + \hat{u}_1 u_2 = 0. \tag{26}$$

The two other equations can be solved explicitly for $n_{1,2}$ in terms of $\mathbf{u}$ and $\hat{\mathbf{u}}$. The substitution of these relations into the equation for the resultant force $\mathbf{n}$ yields a system for $\mathbf{u}$ and n_3 which after elimination of n_3 reads

$$\begin{aligned} & u_1'' u_2 - u_1 u_2'' - u_3' {u_2}^2 - 2\, u_3 u_2' u_2 + \alpha u_2' u_3 u_2 + \alpha {u_2}^2 u_3' \\ & \quad - u_3' {u_1}^2 - 2\, u_1 u_3 u_1' + \hat{u}_1 u_1 u_3' + \alpha u_1 u_1' u_3 + \alpha {u_1}^2 u_3' + \hat{u}_1 {u_3}^2 u_2 = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} & u_2 u_2''' - u_2' u_2'' + u_2 u_1 u_3'' + 2\, u_2 u_3 u_1'' - \hat{u}_1 u_2 u_3'' - \alpha u_2 u_1'' - \alpha u_2 u_1 u_3'' \\ & + 3\, u_2 u_3' u_1' + u_1 u_1' {u_2}^2 + {u_2}^3 u_2' - 2\, u_3 {u_2}^2 u_3' + 2\alpha {u_2}^2 u_3 u_3' u_3 - 2\alpha u_2 u_1' u_3' \\ & - u_2' u_3' u_1 - 2\, u_2' u_3 u_1' + \hat{u}_1 u_2' u_3' + \alpha u_2' u_1' u_3 + \alpha u_2' u_1 u_3' - \hat{u}_1 {u_2}^3 u_3 = 0, \end{aligned} \tag{28}$$

where $\alpha = C/B$ as before. The three last equations form a closed system for the curvature vector $\mathbf{u}$. It is now straightforward to consider the perturbation problem $u_1 = K + \epsilon x$, $u_2 = \epsilon y$ and $u_3 = \epsilon z$, which leads after simplification to the linearized system

$$z''' + \hat{u}_1^2 (1 + \rho\alpha - \rho) z' = 0, \tag{29}$$

with $\rho = K/\hat{u}_1$. Since we are considering a closed rod, the curvature vector must be periodic with period $\frac{2\pi}{K}$. The linearized equation supports periodic solutions of the form $z = \exp(insK)$ for

$$-1 - \rho\alpha + \rho + n^2 \alpha \rho^2 = 0, \tag{30}$$

where n must be an integer for the solution to have the period $\frac{2\pi}{K}$. The solution $n = 0$ corresponds to a rod with constant torsion and curvature, that is a helix which is clearly incompatible with the condition that the rod is closed. The solution $n = 1$ exists only for $\rho = 1$ when the rod is in a stress-free state that is always stable (its energy vanishes). Since the positive roots ρ_n of (30) are strictly decreasing with n, the first non-trivial solution occurs for $n = 2$ for which

$$\rho_2 = \frac{\alpha - 1 + \sqrt{\alpha^2 + 14\alpha + 1}}{8\alpha}. \tag{31}$$

Therefore, we conclude that the instability is triggered when the ring has a radius between 1.68 to 2 times as large as the unstressed radius of curvature (corresponding to $\alpha = 1/2$ and $\alpha = 1$, respectively). The stability analysis of rings with intrinsic curvature has been studied independently by different authors [28, 40, 45–48] and has found some interesting application to the problem of DNA mini-rings [46, 49].

5. Modern Developments

We give here a list of generalizations of Michell's results that have been proposed over the years. This list is not meant to be exhaustive and some important results or applications may have been overlooked.

- **Non-isotropic rods.** The fact that circular rods with arbitrary constant twist exist as a solution of the Kirchhoff equations is a direct consequence of the isotropy and uniformity of the cross-section. That is, for an elastic rod with material properties specified by $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and W, there exist generic circular solutions with arbitrary twist only when the rod is *isotropic* and uniform. A rod is *isotropic* if the bending and shear strains vanish, that is $\hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_2 = \hat{\mathrm{v}}_1 = \hat{\mathrm{v}}_2 = 0$, and the strain-energy function is invariant under rotation about the $\mathbf{d_3}$ axis. It is uniform if its material properties do not change along its length and strains $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ are independent of s. For a non-isotropic rod, the orientation of the principal bending direction with respect to the normal and binormal directions is not arbitrary but only a discrete set of possible orientations is possible [48, 50, 51] (an example of this property is given in the previous section where due to the intrinsic curvature, the isotropy of the cross-section is broken and the orientation of the principal bending direction is fixed).
- **Multicovered rings.** An interesting twist on the problem is to consider rings that are initially multicovered, that is the unstressed state is a filament whose central curve goes q times around the circle and the filament is twisted. Clearly, if contact is taken into account, this starting configuration cannot be obtained. Nevertheless, they are well defined mathematically and are physically relevant as soon as the multicovered rings open and self-contact disappears. Depending on the number q of times the ring is initially covered and the number p of full turns initially in the system, the rod has the topology of a torus knot (p, q) (an example of a (3,2) torus knot is given on Figure 5).
- **Post-buckling analysis and self-contact effect.** What happens to the twisted ring after the instability? Simple experiments seem to suggests that as soon as the critical twist is reached, the ring completely buckles and folds on itself. Without considering the self-contact in the fold position, the solution of the Kirchhoff equations suggest that the ring would completely unfold and fold back periodically into a ring [33]. A weakly nonlinear analysis of its post-buckling behavior shows that for $\alpha < 11/8$, the bifurcation is subcritical and nearby solutions are also unstable. For $\alpha > 11/8$, the bifurcation is supercritical, that is close to the bifurcation, the amplitude of the deformation increases as $\sqrt{Tw - Tw_c}$ [40, 52]. Now, taking into account the self-contact effects that occur when the ring folds back on itself, an entire new class of solution and bifurcation can be uncovered as shown in a beautiful series of papers by Coleman and collaborators [53–55] (see also [56]).
- **Over-twisted rings.** The first unstable mode is $n = 2$ and its spatial configuration is shown in Figure 2. Assuming that the rod is twisted past the critical value, then closed into a ring and released, other modes can be excited. Since different modes start growing in time, the dynamics cannot be neglected and the problem is to establish which mode will be selected in the writhing of the ring. This can be analyzed by considering the maximum of the dispersion relation obtained by perturbation [33]. For instance, for $\alpha = 3/4$ and

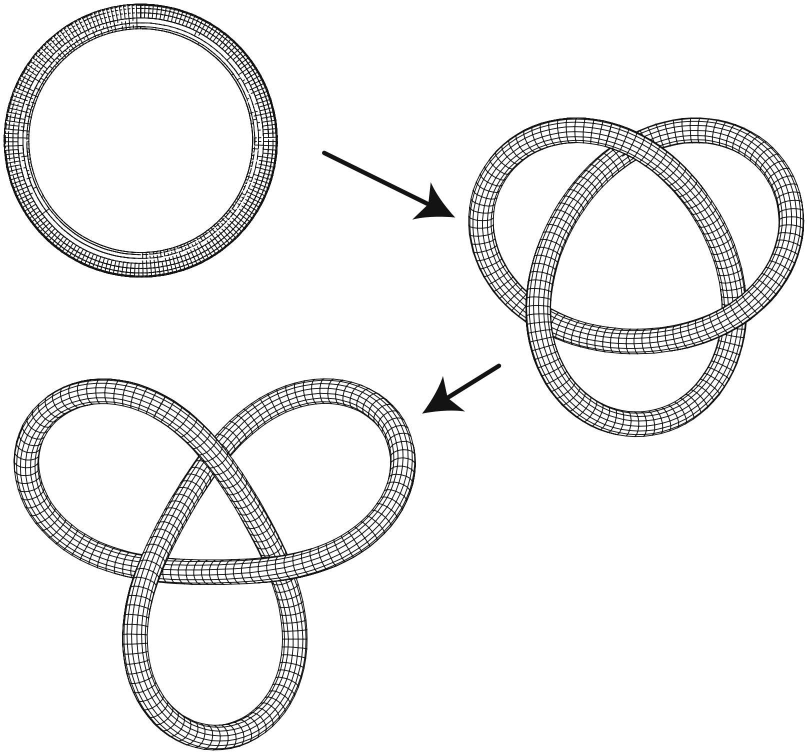

Figure 5 Instability of a multicovered ring. Here $p = 3, q = 2$ and the topology after unfolding is that of the trefoil knot.

$Tw = 30 \gg Tw_c$, all modes between $n = 2$ and $n = 7$ are excited but the fastest growing mode is $n = 4$, leading to the configuration shown in Figure 6.

- **Generalizations.** Further studies and generalizations include the analysis of a growing ring [57], a systematic analysis of all closed loop solutions and their symmetries [44, 58], the stability of a twisted circular ply [59], the statistics of fluctuating rings given by distribution of writhe [60] and the thermal fluctuations of twisted elastic rings with applications to DNA miniplasmids [61].

6. Conclusions

The instability of the twisted elastic ring is a fundamental elastic instability that has been overlooked in most textbooks on elasticity. This is probably due to the fact that the analyses that have been presented require heavy formalisms and lengthy computations that cannot be described succinctly. Michell's century old approach of the problem offers a new way to introduce and discuss the problem in simple terms requiring only the basic equations for the mechanics of rods and the Frenet equations. His analysis should be a cornerstone of three-dimensional elastic instability in rod theory essentially playing the role of Euler Buckling for twisted rods.

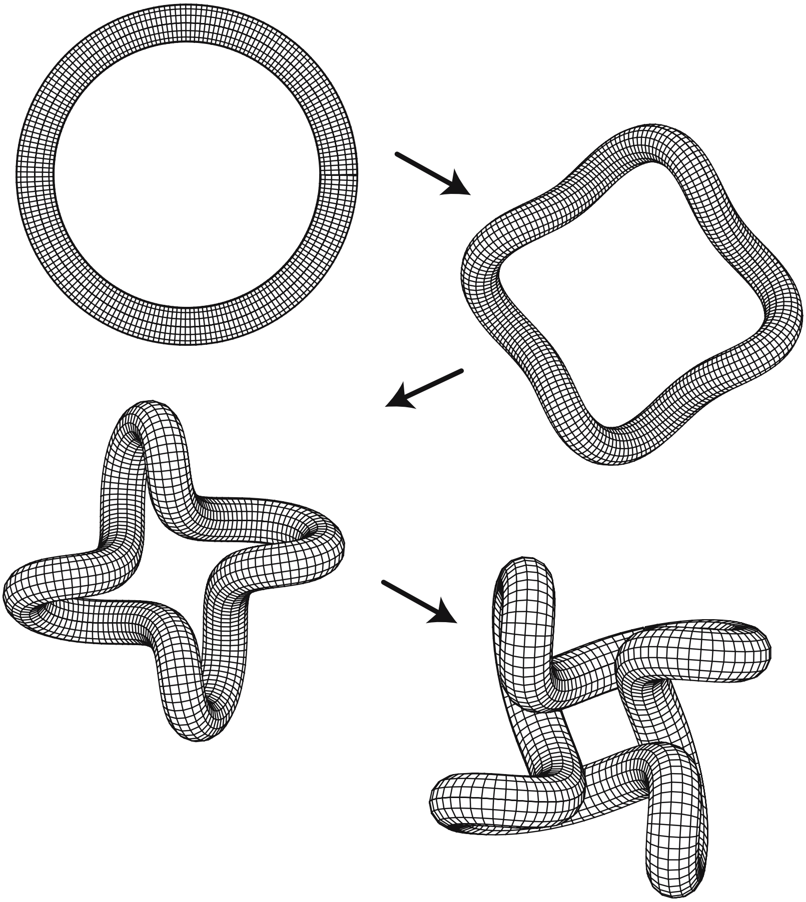

Figure 6 Instability of the overtwisted ring. The twist is $Tw = 30 \gg Tw_c$ and $\alpha = 3/4$, all modes between $n = 2$ and $n = 7$ are unstable and the fastest growing mode is $n = 4$.

Why Michell's work has been forgotten remains a mystery, although it may partly be due to his scientific isolation in Australia. It is also a sign of the new role of mechanics in science. Michell's result has been rediscovered independently at least three times by researchers interested in applications in different fields and well aware of classical results of mechanics. The fact that twisted elastic rings play such an important role in such diverse research areas is a testimony of the central role that elasticity can play today in our understanding of mechanical principles in life and engineering sciences.

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Appendix A

Michell's Original Paper

ON THE STABILITY OF A BENT AND TWISTED WIRE.

If a wire of isotropic section and naturally straight be twisted, and the two ends joined so as to form a continuous curve, the circle will be a stable form of equilibrium for less than a certain amount of twist.*

I propose in this note to determine the limit of stability. I begin by finding the general intrinsic equations of vibration of a bent wire.

Let AB be an element of the wire bounded by normal sections A, B, and let the distances of these sections from a fixed point of the wire be $s - \delta s$ and s respectively.

Let S, T, U be the components of the resultant force on the section B, S being measured along the tangent in the direction of s increasing, T along the principal normal inwards, and U along the binormal, so that the three directions form a right-handed system.

Let F, G, H be the components of the couple on the section B in the same three directions.

Then, if P, Q, R are the impressed forces on the element AB per unit length, the equations of equilibrium are

$$\left.\begin{aligned} \frac{dS}{ds} - \kappa T + P &= 0 \\ \frac{dT}{ds} - \tau U + \kappa S + Q &= 0 \\ \frac{dU}{ds} + \tau T + R &= 0 \\ \frac{dF}{ds} - \kappa G &= 0 \\ \frac{dG}{ds} - \tau H + \kappa F - U &= 0 \\ \frac{dH}{ds} + \tau G + T &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where κ is the curvature and τ the torsion at s.

* Thomson and Tait, *Nat. Phil.*, § 123.

Now let γ be the rate of twist of the wire at s, then the theory of wires gives

$$\left.\begin{array}{l} F = M\gamma \\ G = 0 \\ H = L\kappa \end{array}\right\},$$

assuming the flexibility the same in all directions.

Substituting in equations (1), we get

$$\frac{dF}{ds} = 0,$$

$$U = -L\,\kappa\tau + M\,\kappa\gamma,$$

$$T = -L\frac{d\kappa}{ds},$$

so that the twist γ is constant, and

$$S = -\frac{Q}{\kappa} - \tau\,(L\tau - M\gamma) + L\frac{1}{\kappa}\frac{d^2\kappa}{ds^2}.$$

Substituting in the two remaining equations of (1), we get the two dynamical equations

$$\left.\begin{array}{l} L\dfrac{d}{ds}\left(\dfrac{1}{\kappa}\dfrac{d^2\kappa}{ds^2} + \tfrac{1}{2}\kappa^2 - \tau^2\right) + M\gamma\dfrac{d\tau}{ds} = -P + \dfrac{d}{ds}\dfrac{Q}{\kappa} \\ L\left(\dfrac{d}{ds}\kappa\tau + \tau\dfrac{d\kappa}{ds}\right) - M\gamma\dfrac{d\kappa}{ds} = R \end{array}\right\}\ldots(2).$$

Now let the wire vibrate about its equilibrium form, and let u, v, w be the displacements of the point s along the tangent, principal normal, and binormal respectively at time t.

Supposing no impressed forces we have

$$-P = m\frac{d^2u}{dt^2},$$

$$-Q = m\frac{d^2v}{dt^2},$$

$$-R = m\frac{d^2w}{dt^2},$$

and the condition of inextensibility is

$$\frac{du}{ds} = \kappa v,$$

so that

$$-Q = \frac{m}{\kappa}\frac{d^3u}{ds\,dt^2}.$$

Further, if κ_0, τ_0 denote the equilibrium values of the curvature and torsion respectively, we have*

$$\left.\begin{aligned} \kappa-\kappa_0 &= \frac{d\alpha}{ds}-\tau\beta \\ \tau-\tau_0 &= \frac{d}{ds}\frac{1}{\kappa}\frac{d\beta}{ds}+\kappa\beta+\frac{d}{ds}\frac{\tau}{\kappa}\alpha \end{aligned}\right\} \text{..........(3)},$$

where
$$\alpha=\frac{d}{ds}\frac{1}{\kappa}\frac{du}{ds}+\kappa u-\tau w,$$
$$\beta=\frac{dw}{ds}+\frac{\tau}{\kappa}\frac{du}{ds}.$$

Substituting these values in equations (2), we have the general intrinsic equations of vibration.

When the equilibrium form is a plane curve, these equations reduce to

$$\left.\begin{aligned} L\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2\kappa}{ds^2}+\tfrac{1}{2}\kappa^2\right)+M\gamma\frac{d\tau}{ds} &= m\frac{d^2u}{dt^2}-m\frac{d}{ds}\frac{1}{\kappa^2}\frac{d^3u}{ds\,dt^2} \\ L\left(\frac{d}{ds}\kappa\tau+\tau\frac{d\kappa}{ds}\right)-M\gamma\frac{d\kappa}{ds} &= -m\frac{d^2w}{dt^2} \end{aligned}\right\},$$

where
$$\kappa=\kappa_0+\frac{d^2}{ds^2}\frac{1}{\kappa}\frac{du}{ds}+\frac{d}{ds}\kappa u,$$
$$\tau=\frac{d}{ds}\frac{1}{\kappa}\frac{d^2w}{ds^2}+\kappa\frac{dw}{ds}.$$

Proceeding to the particular case of a circular ring, the equations are

$$L\frac{1}{\kappa^2}\left(\frac{d^3}{ds^3}+\kappa^2\frac{d}{ds}\right)^2u+M\gamma\frac{1}{\kappa}\left(\frac{d^4}{ds^4}+\kappa^2\frac{d^2}{ds^2}\right)w=m\left(\frac{d^2u}{dt^2}-\frac{1}{\kappa^2}\frac{d^4u}{ds^2dt^2}\right),$$
$$-L\left(\frac{d^4}{ds^4}+\kappa^2\frac{d^2}{ds^2}\right)w+M\gamma\frac{1}{\kappa}\left(\frac{d^4}{ds^4}+\kappa^2\frac{d^2}{ds^2}\right)u=m\frac{d^2w}{dt^2}.$$

The appropriate solution, when the wire forms a complete circle, is

$$u=Ae^{i(pt-rs\kappa)},$$
$$w=Be^{i(pt-rs\kappa)},$$

r being an integer.

* "The small deformation of curves and surfaces, &c.," *ante* p. 68.

Making the substitutions and eliminating A, B, we find

$$\begin{vmatrix} mp^2(1+r^2) - L\kappa^4 r^2(1-r^2)^2, & -M\gamma\kappa^3 r^2(1-r^2) \\ -M\gamma\kappa^3 r^2(1-r^2), & mp^2 + L\kappa^4 r^2(1-r^2) \end{vmatrix} = 0,$$

or

$$m^2p^4(1+r^2) + 2mp^2L\kappa^4r^4(1-r^2) - L^2\kappa^8r^4(1-r^2)^3 - M^2\gamma^2\kappa^6r^4(1-r^2)^2 = 0.$$

For stability, the values of p^2 must be positive, and this leads to the condition

$$L^2\kappa^2(r^2-1) > M^2\gamma^2.$$

Now $r=1$ corresponds merely to displacement of the ring as a rigid body.

The necessary condition for stability is therefore

$$\frac{\gamma}{\kappa} < \frac{L}{M}\sqrt{(3)},$$

so that the total twist must be less than

$$2\sqrt{(3)}\,\pi\,L/M.$$

If the cross-section is circular,

$$\frac{L}{M} = \frac{E}{2\mu},$$

where E is Young's modulus and μ is the rigidity modulus.

For metals $E = \frac{5}{2}\mu$ about, and in this case the total twist must be less than $2\pi \times 2.16$.

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