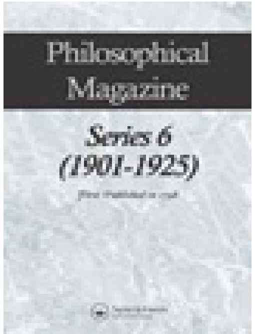

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XXX. *Some Problems of Evaporation.* *By* Harold Jeffreys, *M.A., D.Sc., Fellow of St. John's College, Cambridge*[*].

THE problem of evaporation is practically one of gaseous diffusion: that is to say, if V denote the fraction of the density of the air at any point that is due to water vapour, V will vary from time to time and place to place according to the equation

$$\frac{dV}{dt}=\frac{\partial}{\partial x}\left(k\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial V}{\partial z}\right), \quad . \quad (1)$$

where k is the effective coefficient of diffusion, t the time, and d/dt denotes the total differential following a particle of the fluid. In general V, as above defined, will be referred to as the concentration.

Then if u, v, w be the components of velocity and V be supposed expressed as a function of x, y, z, and t,

$$\frac{dV}{dt}=\frac{\partial V}{\partial t}+u\frac{\partial V}{\partial x}+v\frac{\partial V}{\partial y}+w\frac{\partial V}{\partial z}. \quad . \quad . \quad . \quad (2)$$

The boundary conditions are that the air in contact with a liquid surface is saturated, so that V is there equal to the concentration in saturated air at the temperature of the liquid, and that at a great distance from any liquid V tends to a finite value.

The equation (1) is identical in form with those that determine the transference of heat and momentum; and when the transference is due entirely to turbulence the quantity k has the same value in all three cases[†]. In contact with a solid or liquid surface, on the other hand, the velocity of the medium is zero, and k diminishes to the value it has when there is no turbulence: that is to say, in the evaporation problem k is equal to the coefficient of diffusion of the vapour; in the thermal problem it is the thermometric conductivity of air; and in the equations of motion it is the kinematic viscosity of air. These three quantities are of the same order of magnitude, but are not equal. When the air is at rest k is a constant in all cases and the problem is simply that of solving the equation

$$\frac{dV}{dt}=k\nabla^2 V, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ∇^2 denotes the Laplacian operator.

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[*] Communicated by the Author.

[†] G. I. Taylor, "Eddy Motion in the Atmosphere," Phil. Trans. 215 A. (1915).

When the air is in motion, k at a distance from a boundary is almost wholly due to turbulence, and is practically independent of position. The velocity near a boundary rapidly increases from zero to about half its amount at a considerable distance; this transition is accomplished in a thin layer of shearing, whose thickness in centimetres is estimated at $40/U$, where U is the velocity in centimetres per second at a considerable distance *. In outdoor problems it is therefore usually of the order of a millimetre or smaller. The concentration and temperature also change rapidly within this layer, and in most cases it will be justifiable to assume that at the outer boundary of it each is constant and midway between the values at the surface and at a great distance. Outside this layer transference of heat and vapour will take place according to equation (3), where k is now put equal to the eddy viscosity.

If the dimensions of the liquid surface are of order l, and the time needed for any considerable change of condition over it is of order τ, we see that the terms like $\partial V/\partial t$, $u\partial V/\partial x$, and $k\partial^2 V/\partial x^2$ are relatively of orders

$$1/\tau, \quad U/l, \quad \text{and} \quad k/l^2.$$

Taking k to be of the order of 10^3 cm.^{2}/sec., which is a somewhat small estimate for outdoor problems, $\tau=1$ second, and $U=400$ cm./sec., we see that the first term is small compared with the last provided l is less than 10 cm., and for slower variations this term will be small for still larger values of l. Again, we see that the second term is small compared with the last, provided l is less than 2 cm.; otherwise it must be taken into account. In indoor problems U is probably not far from zero, $k=0{\cdot}24$ cm.^{2}/sec., and the whole of d/dt can therefore be neglected for a surface 1 cm. across provided the saturation near it does not change considerably in 8 seconds.

When l is sufficiently small to satisfy both these conditions the equation of transference reduces to $\nabla^2 V=0$, subject to the same boundary conditions as before. Without loss of generality we can take the concentration at a great distance to be zero. For if it is actually V_d, $V-V_d$ will satisfy the same differential equation and will be constant over all the wetted surfaces, and hence if a problem is solved for $V_d=0$ the solution for any other value of V_d can at once be found by merely writing $V-V_d$ for V throughout.

Now, $\nabla^2 V=0$ is the equation of steady diffusion and

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* Private communication from Major G. I. Taylor.

is also satisfied by the potential in electrostatic problems; it follows that, provided the initial change is not too abrupt and the dimensions are not too great, the value of V at any point is the same as the electric potential at that point when the wetted surface is regarded as a conductor charged to potential V_0, where V_0 is the concentration at the edge of the layer of shearing when the air is moving, and the saturation concentration at the boundary when it is at rest. The charge needed for this is CV_0, where C is the electrostatic capacity of the conductor. Now the rate of transference outwards from the boundary is

$$-\iint k\rho \frac{\partial V}{\partial \nu} dS,$$

where $\partial\nu$ denotes the element of the outward normal, ρ is the density of air, and the integral is taken over the boundary.

But in the electrostatic problem, if σ denotes the density of the electric charge on the surface,

$$\frac{\partial V}{\partial \nu} = -4\pi\sigma,$$

and hence

$$-\iint \frac{\partial V}{\partial \nu} dS = 4\pi \iint \sigma dS = 4\pi C V_0.$$

Hence the rate of transference outwards is

$$4\pi k\rho C V_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This determines the rate of evaporation, which is shown, other things being equal, to be proportional to the linear dimensions, since the electrostatic capacity varies in this way for bodies of the same shape.

The above is practically Stefan's* solution, which has been experimentally proved to be correct subject to the conditions stated †. When the velocity is large enough to need to be taken into account, a general solution is no longer possible, but for a large wetted surface the curvature may be neglected, and the problem reduces to that of a wind blowing over a flat surface. This is treated in the next section.

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* *Wien. Akad. Ber.* lxxxiii. Abteil 2, p. 613 (1881).

† H. T. Brown & F. Escombe, Phil. Trans. 193 B. pp. 223-291.

Steady wind blowing over a flat surface of water.

Take the plane of the surface for that of $z=0$. Let u be the velocity of the wind in the direction of the x axis. There is no wind in the directions of the other two axes. Then outside of the layer of rapid shearing the equation of diffusion when a steady state has been attained is

$$u\frac{\partial \mathrm{V}}{\partial x}=k\frac{\partial^2 \mathrm{V}}{\partial z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

provided the distance from the margin is great enough for $\frac{\partial^2 \mathrm{V}}{\partial x^2}$ and $\frac{\partial^2 \mathrm{V}}{\partial y^2}$ to be neglected in comparison with $\frac{\partial^2 \mathrm{V}}{\partial z^2}$. The velocity u is a function of z only, and k/u is supposed constant, equal to h^2, say. The value of V over the surface when x is positive is V_0. Then all the conditions are satisfied if

$$\mathrm{V}=\mathrm{V}_0\left(1-\operatorname{Erf}\frac{zx^{-\frac{1}{2}}}{2h}\right) \text{ when } x \text{ is positive.} \quad . \quad (6)$$

$$\mathrm{V}=0 \text{ when } x \text{ is negative.}$$

This makes $\frac{\partial \mathrm{V}}{\partial z}=\frac{\mathrm{V}_0}{h\sqrt{(\pi x)}}$ over the wet surface.

Hence the rate of evaporation is

$$k\rho\frac{\partial \mathrm{V}}{\partial z}=\mathrm{V}_0\rho\sqrt{\frac{ku}{\pi x}} \text{ per unit area.} \quad . \quad . \quad . \quad (7)$$

Then the amount evaporated between 0 and x over a strip dy in diameter is by integration

$$2\rho\mathrm{V}_0(kux/\pi)^{\frac{1}{2}}dy.$$

Finally, if the length of the strip from one margin to the other be l, and parts near enough to the edges for the end corrections at both ends to be important be neglected, the amount evaporated can be found at once to be

$$2\rho\mathrm{V}_0\left(\frac{ku}{\pi}\right)^{\frac{1}{2}}\int l^{\frac{1}{2}}dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

taken over the whole area.

Hence for areas of the same shape and linear dimensions proportional to a the rate of evaporation would be proportional to $a^{1\cdot5}$. In particular, for a circular area of radius a it would be

$$3{\cdot}95\,\rho\mathrm{V}_0(kua^3)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It has been assumed in this treatment that the linear dimensions are so large that $\frac{\partial^2 V}{\partial x^2}$ and $\frac{\partial^2 V}{\partial y^2}$ are negligible in comparison with $\frac{\partial^2 V}{\partial z^2}$.

Now the value of V in (6) makes the first two quantities of order V_0/a^2 and the last of order V_0/h^2a. The condition required for the method to be justifiable is therefore that a shall be large compared with h^2; in other words, ua/k shall be large.

It has also been assumed that k/u is constant. This requires that the distance out to which the disturbance extends shall be sufficiently small for the change in k/u from its value just outside a thin surface-layer to be small compared with the whole. This distance is of order $2ha^{\frac{1}{2}}$. Thus $2a^{\frac{1}{2}}dh/dz$ must be small. Now $\frac{1}{h}\frac{dh}{dz}$ is probably of the same order as $\frac{1}{u}\frac{du}{dz}$. But $k\rho\frac{du}{dz}$ when $z=0$ is about $\cdot 003\rho u^2$, for each quantity is equal to the skin friction per unit area. Thus $\cdot 006(ua/k)^{\frac{1}{2}}$ must be small. This method is therefore correct provided ua/k lies between, say, 4 and 10^4.

Consider first an outdoor case, with $u=400$ cm./sec., $k=1000$ cm.^{2}/sec. Then a must be between 10 cms. and 250 metres. In a room with little draught, we may have $u=4$ cm./sec., $k=1$ cm.^{2}/sec., and a must be between 1 cm. and 25 metres.

When ua/k is much less than unity, the velocity of the wind may be neglected and the problem becomes one of steady diffusion; the rate of evaporation is then proportional to the linear dimensions. When ua/k is large compared with 10^4, the effect will extend upwards for such a distance that in most of the volume concerned u and k will be nearly constant and equal to their values at a considerable height. The air up to such a height will be practically saturated, and vapour will diffuse from the upper surface of the saturated air precisely as before; the law that the rate of evaporation is proportional to $a^{1\cdot 5}$ will therefore still hold, with different values of u and k.

The result that for bodies of medium dimensions the rate of evaporation is practically proportional to $a^{1\cdot 5}$ has been discovered experimentally by Thomas and Ferguson*, who also point out that this gives a fair representation of Renner's †

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* N. Thomas and A. Ferguson, Phil. Mag. xxxiv. pp. 308–321 (1917).

† O. Renner, *Flora*, 100. p. 474 (1910).

results. Brown and Escombe also found that the actual rate of evaporation varied more rapidly than the radius but not so rapidly as the area. Thomas and Ferguson find that when the surface of the water is below the rim of the containing vessel and the evaporation is assumed proportional to a^n, where n is a constant for a given depth, n tends to 2 when the depth is large.

The evaporation from a circular cylinder, wet at the bottom and open at the top, may be considered here. So long as the depth is either great or small compared with the radius, we can assume that within the cylinder the layers of equal concentration are parallel to the base. Let l be the depth, a the radius, and R the rate of evaporation. Let the concentration at the bottom be V_0, and at the top V_1. Then inside the cylinder the condition for steady flow makes

$$R = \pi a^2 k\rho \frac{\partial V}{\partial z} = \pi a^2 k\rho \frac{V_0 - V_1}{l}.$$

Outside the cylinder, if the air is at rest, the vapour is practically diffusing from one side of a disk at concentration V_1, and therefore the rate of evaporation is $4k\rho a V_1$ * ; this also must be equal to R, else the concentration at the mouth would be changing. Eliminating V_1 we find at once

$$R = \pi a^2 k\rho V_0/(l + \tfrac{1}{4}\pi a),$$

agreeing with a result of Brown and Escombe (*loc. cit.* p. 258).

If a strong wind is blowing over the top, the conditions inside will be unaltered, but at the top we shall have

$$R = 3{\cdot}95\, \rho V_1 (kua^3)^{\frac{1}{2}},$$

giving, after elimination of V_1,

$$R = \rho V_0 \div \left(\frac{l}{\pi a^2 k} + \frac{1}{3{\cdot}95\,(kua^3)^{\frac{1}{2}}}\right),$$

which varies with l in the way found by Thomas and Ferguson.

Evaporation from the surface of a leaf.

The surface of a leaf consists of an almost impermeable cuticle perforated by a large number of small holes, called stomata, through which respiration and absorption of carbon dioxide take place. It is a matter of some uncertainty

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* The capacity of a circular disk is $2a/\pi$, and therefore that of one side of it is a/π.

whether carbon dioxide can by simple diffusion enter these holes as fast as is indicated by the observed rate of carbon assimilation. If the rate of absorption were proportional to the area of the holes it could not be great enough, but if each hole absorbed at a rate proportional to its radius (this being the correct law for an isolated hole as small as a stoma) the total absorption would be much greater than is required. Some points in the theory still require examination; it is not obvious that the surrounding stomata will not interfere with the action of any individual to an important extent, and a wind blowing over the surface, though unimportant for a single stoma, may be important when there are thousands of them spread over a considerable area. The problem is mathematically the same as that of evaporation from the stomata, by which it will be replaced.

First, consider the leaf to be in a steady state and wind absent. Let the radius of a stoma be a, and the number per unit area n^2. Then the average distance between stomata is of order $1/n$, and is large compared with a. The value of V over the surface of any stoma is V_0. Then at a distance r from an isolated stoma V is of order V_0a/r. Now, if the stomata acted independently of one another, consider some particular stoma. By itself it would make $V=V_0$ over it; the others will together add to this an amount $\Sigma\frac{V_0a}{r}$, which is not very different from $\iint\frac{V_0a}{r}n^2dS$, taken over the whole surface of the leaf. This is of order $2\pi V_0an^2l$, where l is of the order of the dimensions of the leaf. Now Jörgensen and Stiles give* for a typical case $2a=0{\cdot}00107$ cm.; $n^2=33{,}000/\text{cm.}^2$ Thus the addition to V by the neighbouring stomata would be of the order of 300 V_0 for a leaf of radius 3 cm. This is of course impossible, for V cannot be greater than V_0. The meaning of the result is that the surroundings are enough to cause the air at any point to be practically saturated, and only a small portion of the vapour-pressure over any stoma is maintained by that stoma itself. The total evaporation from the surface of a leaf is therefore the same as would take place if V were equal to V_0 over the whole surface, and its amount is therefore $4\pi kCV_0$, where C is the electrostatic capacity of the whole surface of the leaf.

Looking at the matter in another way, the rate of evaporation from a single stoma uninfluenced by its surroundings

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* 'Carbon Assimilation,' p. 63.

would be $2\pi k c V_0$ [*], where c is the electrostatic capacity of a disk of the dimension of the stoma, and from all the stomata on a leaf of area A it would be $2\pi k n^2 c V_0 A$, if they did not influence one another. On the other hand, the evaporation in unit time from the whole surface of a leaf when completely wet is $4\pi k C V_0$. The ratio of these two rates is of order $n^2 a A/2l$, or practically $n^2 al$. Now, taking l to be 3 cm., and $a = 0{\cdot}00053$ cm., it follows that $n^2 al$ is unity if n^2 is as large as 600. Hence, if there are more than 600 stomata per square centimetre, the rate of evaporation from them will be greater than that from the whole surface of a wet leaf, which is absurd. It follows that evaporation must be enormously restricted by the presence of other stomata. Brown and Escombe have stated that "the interference of the density shells of small holes set at 10 diameters or more apart is small, each hole beyond this limit acting almost independently according to the diameter law." The above result shows that this is erroneous [†]; in fact, the authors themselves imply its error in the diagram they give of the lines of flow through a multiperforate septum, when they make them become approximately parallel at a short distance from the septum.

Again, taking $n^2 = 33000/\text{cm.}^2$ and $l = 3$ cm., we see that $n^2 al$ is greater than unity unless a is less than 10^{-5} cm. Hence we shall still have the result that the evaporation is practically independent of the opening of the stomata as long as a exceeds this limit; in other words, until the stomata contract to $\frac{1}{50}$ of their original diameter, the rate of evaporation will be practically independent of the diameter [‡]; when they have closed still further it will decrease, and will finally vanish when they are quite closed. This appears to contain the answer to Sir F. Darwin's criticism [§] of some results of Lloyd [‖]. Lloyd has stated that the regulatory function of stomata is almost *nil*, which Sir F. Darwin

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[*] We must have 2 instead of 4, as evaporation can only take place from one side of the stoma.

[†] In Brown and Escombe's experiments with multiperforate septa the radius a was 0·019 cm., n^2 was 100 to 2·77, and l about 2 cm., making $n^2 al$ equal to about 4 at the most, and in most cases much smaller. The results are therefore irrelevant to the case of a leaf.

[‡] Brown and Escombe, Phil. Trans. 193 B. p. 278 (1900), say that stomata can close to $\frac{1}{15}$ of their diameter without affecting assimilation.

[§] "The Relation between Transpiration and Stomatal Aperture," Phil. Trans. 207 B. p. 413 (1915–16).

[‖] F. E. Lloyd, "The Physiology of Stomata," Carnegie Institution, 1908.

regards as inconsistent with his further results that "complete closure . . . . . reduces transpiration to or nearly to cuticular rate," and "when the stomata are open to their utmost limit the highest rate of transpiration is the maximum of which the leaf is capable." A more satisfactory statement would be that until the stomatal aperture is reduced to a certain very small value the possible rate of transpiration is practically independent of the aperture, and nearly all of the reduction to zero when the stoma closes takes place in the last 2 per cent. of the reduction of aperture.

Next, consider the effect of wind. Suppose for simplicity that the stomata are arranged in straight rows, the distance between consecutive rows and between consecutive stomata on the same row being b. Then $b=1/n$. Consider a square column of air of side b. To pass over a stoma it would take a time b/u, and if it were unsaturated at the commencement it would therefore acquire a weight of vapour $2\pi k\rho c\mathrm{V}_0 b/u$, if there were no mutual influence between stomata. Now suppose the air to have moved forward a distance x, in time x/u. Then the vapour in it will have spread out by diffusion through a radius comparable with $2(kx/u)^{\frac{1}{2}}$; and if x/b is great diffusion parallel to the surface of the leaf and across the wind will have practically ceased, and thus the vapour will occupy half a flat cylinder of radius $2(kx/u)^{\frac{1}{2}}$ and thickness b, its centre being of course at the point x. Thus the concentration in it will be of order $c\mathrm{V}_0/x$. Further, the number of stomata much affected will be of order

$$4n^2b(kx/u)^{\frac{1}{2}}=4n(kx/u)^{\frac{1}{2}}.$$

Similarly, the number of stomata whose influence at this time will have affected the column of air when it has travelled a distance between $x-\frac{1}{2}b$ and $x+\frac{1}{2}b$ is $4n(kx/u)^{\frac{1}{2}}$, and therefore the total concentration produced by them is $4nc\mathrm{V}_0(k/ux)^{\frac{1}{2}}$. This is then the concentration acquired by a mass of air on account of what happened between times $(x\pm\frac{1}{2}b)/u$ previously, and the total produced by *all* times is to be found by summing the series for all such intervals. Put $x/b=r$. The total is then $4nc\mathrm{V}_0(k/ub)^{\frac{1}{2}}\Sigma r^{-\frac{1}{2}}$, the summation being from $r=1$ to $r=l/b$, where l is the distance of the mass from the stoma nearest the margin. When l is great this is of the order of $8n^2c\mathrm{V}_0(kl/u)^{\frac{1}{2}}$. Now b is about 0·05 mm., and thus is usually small compared with the thickness of the layer of rapid shearing *. *A fortiori* a, and hence c, are

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* See p. 271.

smaller. Thus u, being the velocity within the region to which the diffusion from a stoma extends, is much less than the velocity outside. Similarly k is the true coefficient of diffusion, about 0·24 cm.^{2}/sec. The thickness of the layer of shearing being 40/U, it follows that u is of order $U^2a/40$. Thus the quantity just obtained is of order $100n^2V_0(kla)^{\frac{1}{2}}/U$, which is about 100 V_0. It follows by argument similar to that used in the case of no wind that the earlier stomata saturate the air before the later ones are reached. Thus the total evaporation is not very different from that in the case where the whole surface of the leaf is wet*, and is therefore proportional to $l^{1·5}$, where l is now proportional to the linear dimensions of the leaf.

This approximation will break down if n^2a is much smaller, for then the residual saturation from the earlier stomata may be small compared with V_0, and the rate of evaporation will then be the same as that obtained by summing the results for the individual stomata, each being supposed isolated; this sum is $2\pi n^2kpcV_0A$. A similar result may be obtained if u/k is much greater. It must always be noted, however, that this formula can be applied only when the result it gives is less than the rate of evaporation from a wet leaf with the wind blowing over it; otherwise we should again have the absurdity of the evaporation from a part being greater than that from the whole.

The best method of determining whether it is better to employ the sum of the possible evaporations from the individual stomata, or to regard the whole surface of the leaf as wet, is probably to calculate the rate of evaporation on both bases and take the smaller of the two results as supplying the correct upper limit to the amount of respiration the leaf can perform. Similar remarks will apply to the possible absorption of substances from the air.

It may be remarked that when the number of stomata is so large as to make the problem reduce to that of a wet leaf, the total evaporation is not a function of the number of stomata, but that from any single stoma is inversely proportional to the number. Thus increasing the number diminishes the work thrown on any individual, which may be of some physiological importance.

The above investigation concerns only the purely physical

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* O. Renner, *Flora*, vol. 100. pp. 451–547 (1910), states on p. 485 that the evaporation from a leaf is the same as that from a water surface

side of diffusion. It does not preclude the possibility that a reduction of the stomatal aperture may be associated with a reduction of the rate of evaporation ; but it does show that in most cases the cause of such reduction is not the mere extra mechanical obstruction to the passage of water vapour, but must depend on the internal conditions. The importance of these is obvious. For instance, in the problem considered here the air has been supposed saturated when in contact with a stoma and perfectly dry at a great distance. Actually the concentration at a great distance has the finite value V_d ; and that at a stomatal aperture is probably somewhat less than the saturation concentration. Let it be V_1. The latter question is further complicated by the facts that the dissolved substances within the cells must diminish the pressure of saturated vapour ; that the leaf is normally at a somewhat higher temperature than its surroundings, so that the pressure of saturated vapour will on this account be greater than that at the temperature of the surroundings ; and that owing to internal restrictions to the supply of water to the stoma the vapour-pressure may be reduced. The effect of these changes is that in all the formulæ we must substitute $V_1 - V_d$ for V_0.

Another complication arises from the fact that the stomata are not usually mere pores with saturated air in their planes ; in most cases they are pits sunk in the leaf-surface. As long as their number is large this is not likely to produce any great effect on the rate of evaporation, for in exactly the same way as with flat stomata the earlier ones met by the air will partially saturate it, and the air when it meets the later ones will be nearly at the same saturation as that *inside* them. When the number is small, on the other hand, the formula for evaporation from depressed stomata must be used. For circular cylindrical stomata this gives for the rate of evaporation when the depth l is great compared with the radius

$$\frac{\pi n^2 k \rho a^2 (V_1 - V_d) A}{l + \frac{1}{4}\pi a}.$$