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Time crystals: Can diamagnetic currents drive a charge density wave into rotation?

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Abstract – It has been recently argued that an inhomogeneous system could rotate spontaneously in its ground state —hence a "time crystal" which is periodic in time. In this letter we present a very simple example: a superfluid ring threaded by a magnetic field which develops a charge density wave (CDW). A simple calculation shows that diamagnetic currents cannot drive rotation of the CDW, with a clear picture of the cancellation mechanism.

In a recent letter [1] Frank Wilczek introduced a revolutionary concept, "time crystals" which in their ground state would be time dependent. A simple example is a ring threaded by a magnetic field that breaks time-reversal invariance. If a charge density wave (CDW) appears, will diamagnetic currents put it in rotation? Such a challenging proposal raised a vivid controversy [2], especially with my colleague Patrick Bruno who recently provided a general proof that it is impossible [3]. In order to clarify the underlying physics, I study here a very simple model in which elementary calculations can be done explicitly from beginning to end.

Consider a circular ring with radius R, perimeter $L = 2\pi R$, threaded by a magnetic flux Φ. The vector potential along the ring is $A = \Phi/L$. As a model take a Bose condensate of charge q particles (for Cooper pairs $q = 2e$) with a density $n_0 = N_0/L$. The superfluid phase is S, the current density on the ring and the energy are

$$J = \frac{n_0}{m}\left[\hbar\,\mathrm{grad}\,S - \frac{qA}{c}\right], \quad E_0 = \frac{N_0}{2m}\left[\hbar\,\mathrm{grad}\,S - \frac{qA}{c}\right]^2.$$

The circulation of $\mathrm{grad}\,S$ is quantized, equal to $2\pi\nu$, where ν is an integer. The linear momentum per particle is $p = \hbar\,\mathrm{grad}\,S$, without the gauge term. The angular momentum per particle is

$$L_z = R\hbar\,\mathrm{grad}\,S = R\hbar\frac{2\pi\nu}{2\pi R} = \nu\hbar.$$

We recover the usual quantization of angular momentum.

Consider first a weak magnetic field whose flux is smaller than half a quantum: the ground state corresponds to $\nu = 0$, with a diamagnetic current $J = -n_0 qA/mc$. Assume now that the ring presents a spontaneous charge density wave with a density modulation n_1. We do not need a periodic n_1, we only request a modulation, $\oint n_1 \mathrm{d}x = 0$. The current equation becomes

$$J = \left(\frac{n_0 + n_1}{m}\right)\left[\hbar\,\mathrm{grad}\,S - \frac{qA}{c}\right].$$

J is conserved and $\mathrm{grad}\,S$ cannot vanish. We must solve the equation

$$\hbar\,\mathrm{grad}\,S = \frac{qA}{c} + \frac{mJ}{n_0 + n_1}$$

with the two conditions $\oint \mathrm{grad}\,S\mathrm{d}x = 0$ and $\oint n_1\mathrm{d}x = 0$ which will fix the modulation $\mathrm{grad}\,S$ and the unknown current J. Integrating that equation over the ring we find

$$\frac{qA}{c} + mJ\overline{\left(\frac{1}{n}\right)} = 0.$$

Note that regions of small n severely reduce the current J, a feature pointed out long ago by Tony Leggett [4]. Here n_1 is small and in lowest order we find

$$J = -\frac{n_0 qA}{mc}\left[1 - \frac{\overline{n_1^2}}{n_0^2}\right].$$

The charge density wave reduces the diamagnetic current by an amount of order $\overline{n_1^2}$. The ground-state energy may be written as

$$E_0 = \frac{1}{2}\oint J\left[\hbar\,\mathrm{grad}\,S - \frac{qA}{c}\right] = -\frac{qAJ}{2c} = \frac{N_0 q^2 A^2}{2mc^2}\left[1 - \frac{\overline{n_1^2}}{n_0^2}\right].$$

Note that the charge density wave *reduces* the energy, a feature that should eventually be added to the usual Landau picture for the appearance of n_1,

$$E_{CDW} = -\alpha n_1^2 + \beta n_1^4 \quad\longrightarrow\quad n_1^2 = \frac{\alpha}{2\beta}.$$

Diamagnetism *enhances* the charge density instability.

Finally, the angular momentum L_z, zero for a diamagnetic current in a perfect ring, does not vanish any more when the density wave appears:

$$L_z = \oint (n_0 + n_1)\, R\hbar\,\mathrm{grad}\,S = \oint R\left[(n_0+n_1)\frac{qA}{c} + mJ\right] = N_0 R\frac{qA}{c}\frac{\overline{n_1^2}}{n_0^2}.$$

Hence a crucial question: could such an angular momentum induce a spontaneous rotation of the charge density wave? Standard wisdom says that rotating the frame at angular velocity Ω adds a term $L_z\Omega$ to the energy: if true, rotation of the charge density wave is unavoidable since all other terms in the energy are quadratic in Ω. A wise pedestrian approach is to stay in the laboratory frame when calculating energy, noting, however, that in that frame the current is no longer conserved. It is conserved in the rotating frame of the charge density wave: we start from that statement and we bring everything back to the laboratory frame.

Let J' be the constant current in the rotating frame. The previous calculation relates it to a vector potential A' unknown as of now,

$$J' = \left(\frac{n_0+n_1}{m}\right)\left[\hbar\,\mathrm{grad}\,S - \frac{qA'}{c}\right] \Longrightarrow \hbar\,\mathrm{grad}\,S = \frac{qA'}{c} + \frac{mJ'}{n_0+n_1}.$$

The current J in the laboratory frame is

$$J = J' + (n_0+n_1)\,\Omega R = \left(\frac{n_0+n_1}{m}\right)\left[\hbar\,\mathrm{grad}\,S - \frac{qA}{c}\right].$$

It follows that $A' = A + mc\Omega R/q$: J' is related to A' exactly as J was to A before. Rotating the charge density wave is tantamount to replacing the vector potential A by A' in the rotating frame. From there on the calculation unfolds as before. The energy in the laboratory frame may be written as

$$\begin{aligned} E_0 &= \oint \frac{n_0+n_1}{2m}\left[\hbar\,\mathrm{grad}\,S - \frac{qA}{c}\right]^2 \\ &= \oint \frac{n_0+n_1}{2m} \\ &\times \left[\left(\hbar\,\mathrm{grad}\,S - \frac{qA'}{c}\right)\left(\hbar\,\mathrm{grad}\,S - \frac{qA''}{c}\right) + m^2\Omega^2R^2\right] \\ &= \frac{1}{2}\oint\left[J'\left(\hbar\,\mathrm{grad}\,S - \frac{qA''}{c}\right) + n_0 m\Omega^2 R^2\right], \end{aligned}$$

where we have set $A'' = A - mc\Omega R/q$. The constant current J' is

$$J' = -\frac{n_0 qA'}{mc}\left[1 - \frac{\overline{n_1^2}}{n_0^2}\right].$$

We find the energy

$$E_0 = \frac{N_0}{2m}\oint\left[\frac{q^2A'A''}{c^2}\left(1 - \frac{\overline{n_1^2}}{n_0^2}\right) + m^2\Omega^2R^2\right].$$

The correction due to rotation is of order Ω^2. There is no linear term that could generate spontaneous rotation. Note that this second-order term vanishes if $n_1 = 0$: rotating something which does not exist cannot cost any energy! In contrast, rotating the density wave costs an energy,

$$E_1 = \frac{mN_0}{2}\Omega^2R^2\frac{\overline{n_1^2}}{n_0^2}.$$

The conclusion of this naive model is clear: *a charge density wave is not driven to rotation by a diamagnetic current in the ground state* $\nu = 0$.

Generalization to an excited state $k = \nu/2\pi R$ is straightforward. We still have

$$\hbar\,\mathrm{grad}\,S = \frac{qA}{c} + \frac{mJ}{n_0+n_1}$$

whose circulation is

$$\frac{\hbar\nu}{R} = \frac{qA}{c} + \frac{mJ}{n_0}\left[1 + \frac{\overline{n_1^2}}{n_0^2}\right].$$

The constant current J and the energy become

$$J = \frac{n_0}{m}\left[1 - \frac{\overline{n_1^2}}{n_0^2}\right]\left[\frac{\hbar\nu}{R} - \frac{qA}{c}\right],\quad E = \frac{1}{2}\oint J\left[\hbar\,\mathrm{grad}\,S - \frac{qA}{c}\right].$$

Since J is constant only the circulation of $\mathrm{grad}\,S$ matters, hence

$$E = \frac{N_0}{2m}\left[\frac{\hbar\nu}{R} - \frac{qA}{c}\right]^2\left[1 - \frac{\overline{n_1^2}}{n_0^2}\right].$$

The only difference is the replacement of eA/c by $(eA/c-\hbar\nu/R)$. From there on the calculation is unchanged.

Our final conclusion is clear, but limited: a charge density wave is not driven to rotation in a quantum coherent state, for instance by a diamagnetic current induced by a magnetic field, or by a persistent current in a coherent, phase locked, superfluid state. This is consistent with the Ehrenfest theorem which states that the expectation value of the time derivative of any observable A is identically zero in any eigenstate $|\psi_n\rangle$ of the Hamiltonian:

$$\left\langle \frac{\mathrm{d}A}{\mathrm{d}t} \right\rangle = i \langle\psi_n|\, AH - HA\, |\psi_n\rangle = 0.$$

Any local motion of the charge density wave creates a local time dependence which is precluded. The ring is a finite system and CDW motion is a local issue. Such a conclusion holds for the ground state as well as for thermal equilibrium where the density matrix is diagonal in the $|\psi_n\rangle$ basis. This is no longer true if a current is forced in the ring, breaking thermal equilibrium. Then CDW dragging becomes possible.

* * *

I wish to thank my colleagues Patrick Bruno and Andres Cano who introduced me to the challenge of time crystals. Numerous discussions with them were crucial in my search for simplicity. The reader is referred to the paper of Patrick Bruno [3] which offers a much more general proof, necessarily more elaborate. His conclusions are fully consistent with my simple picture.

Additional remark: Nowadays basic concepts and simple proofs may hardly find their way on highly reputed physics journals. That was the fate of the present letter, which I dedicate to young physicists who conceive original ideas and strive to get their work published.

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