

# Mathematical Tripos Part IB: Lent 2018

## Numerical Analysis – Lecture 7<sup>1</sup>

### 4.2 Multistep methods

It is often useful to use past solution values in computing a new value to the ODE (4.1). Assuming that  $\mathbf{y}_n, \mathbf{y}_{n+1}, \dots, \mathbf{y}_{n+s-1}$  are available, where  $s \geq 1$ , we say that

$$\sum_{l=0}^s \rho_l \mathbf{y}_{n+l} = h \sum_{l=0}^s \sigma_l \mathbf{f}(t_{n+l}, \mathbf{y}_{n+l}), \quad n = 0, 1, \dots, \quad (4.4)$$

where  $\rho_s = 1$ , is an  $s$ -step method. If  $\sigma_s = 0$ , the method is *explicit*, otherwise it is *implicit*. If  $s \geq 2$ , we need to obtain extra *starting values*  $\mathbf{y}_1, \dots, \mathbf{y}_{s-1}$  by different time-stepping method.

*Examples:* The following are some common multistep methods:

$$\begin{aligned} \text{Euler: } & \mathbf{y}_{n+1} - \mathbf{y}_n = h \mathbf{f}(t_n, \mathbf{y}_n), \\ \text{Implicit Euler: } & \mathbf{y}_{n+1} - \mathbf{y}_n = h \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}), \\ \text{Trapezoidal rule: } & \mathbf{y}_{n+1} - \mathbf{y}_n = \frac{1}{2} h [\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})] \\ \text{Theta rule: } & \mathbf{y}_{n+1} - \mathbf{y}_n = h [\theta \mathbf{f}(t_n, \mathbf{y}_n) + (1 - \theta) \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})] \\ \text{2-step Adams-Bashforth: } & \mathbf{y}_{n+2} - \mathbf{y}_{n+1} = h [\frac{3}{2} \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - \frac{1}{2} \mathbf{f}(t_n, \mathbf{y}_n)] \\ \text{2-step Adams-Moulton: } & \mathbf{y}_{n+2} - \mathbf{y}_{n+1} = h [\frac{5}{12} \mathbf{f}(t_{n+2}, \mathbf{y}_{n+2}) + \frac{2}{3} \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - \frac{1}{12} \mathbf{f}(t_n, \mathbf{y}_n)]. \end{aligned}$$

For example Adams-Bashforth is a 2-step method ( $s = 2$ ) with  $\rho_2 = 1, \rho_1 = -1, \rho_0 = 0$  and  $\sigma_2 = 0, \sigma_1 = \frac{3}{2}$  and  $\sigma_0 = -\frac{1}{2}$ . The implicit Euler method, trapezoidal rule, theta rule for  $0 \leq \theta < 1$ , and Adams-Moulton are *implicit* methods. The reason these are called implicit is that  $\mathbf{y}_{n+s}$  appears in the right-hand side of (4.4) and so one has to solve a (generally nonlinear) algebraic equation to compute the new value  $\mathbf{y}_{n+s}$  from the recursion rule.

Our goal is to develop some general tools to study the convergence of multistep methods. We first introduce the definition or *order*.

**Order:** The *order* of the multistep method (4.4) is the largest integer  $p \geq 0$  such that

$$\sum_{l=0}^s \rho_l \mathbf{y}(t_{n+l}) - h \sum_{l=0}^s \sigma_l \mathbf{y}'(t_{n+l}) = \mathcal{O}(h^{p+1}) \quad (4.5)$$

for all sufficiently smooth functions  $\mathbf{y}$ . The order is a local measure of accuracy for the method: it measures the error incurred by applying the rule (4.4), assuming that the correct value of  $\mathbf{y}$  at the previous points is known. Let us evaluate the order of some of the methods given above:

**The order of Euler's method:** For Euler's method, the left-hand side of (4.5) is

$$\mathbf{y}(t_{n+1}) - [\mathbf{y}(t_n) + h \mathbf{y}'(t_n, \mathbf{y}(t_n))] = [\mathbf{y}(t_n) + h \mathbf{y}'(t_n) + \frac{1}{2} h^2 \mathbf{y}''(t_n) + \dots] - [\mathbf{y}(t_n) + h \mathbf{y}'(t_n)] = \mathcal{O}(h^2)$$

and we deduce that Euler's method is of order 1.

**The order of the theta method:** From Taylor's theorem we have:

$$\begin{aligned} & \mathbf{y}(t_{n+1}) - \mathbf{y}(t_n) - h [\theta \mathbf{y}'(t_n) + (1 - \theta) \mathbf{y}'(t_{n+1})] \\ &= [\mathbf{y}(t_n) + h \mathbf{y}'(t_n) + \frac{1}{2} h^2 \mathbf{y}''(t_n) + \frac{1}{6} h^3 \mathbf{y}'''(t_n)] - \mathbf{y}(t_n) - \theta h \mathbf{y}'(t_n) \\ & \quad - (1 - \theta) h [\mathbf{y}'(t_n) + h \mathbf{y}''(t_n) + \frac{1}{2} h^2 \mathbf{y}'''(t_n)] + \mathcal{O}(h^4) \\ &= (\theta - \frac{1}{2}) h^2 \mathbf{y}''(t_n) + (\frac{1}{2} \theta - \frac{1}{3}) h^3 \mathbf{y}'''(t_n) + \mathcal{O}(h^4). \end{aligned}$$

---

<sup>1</sup>Corrections and suggestions to these notes should be emailed to [h.fawzi@damtp.cam.ac.uk](mailto:h.fawzi@damtp.cam.ac.uk).

Therefore the theta method is of order 1, except that the trapezoidal rule ( $\theta = 1/2$ ) is of order 2.

Let  $\rho(w) = \sum_{l=0}^s \rho_l w^l$ ,  $\sigma(w) = \sum_{l=0}^s \sigma_l w^l$ .

**Theorem** *The multistep method (4.4) is of order  $p \geq 1$  iff*

$$\rho(e^z) - z\sigma(e^z) = \mathcal{O}(z^{p+1}), \quad z \rightarrow 0. \quad (4.6)$$

**Proof.** Substituting the exact solution and expanding into Taylor series about  $t_n$ ,

$$\begin{aligned} \sum_{l=0}^s \rho_l \mathbf{y}(t_{n+l}) - h \sum_{l=0}^s \sigma_l \mathbf{y}'(t_{n+l}) &= \sum_{l=0}^s \rho_l \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{y}^{(k)}(t_n) l^k h^k - h \sum_{l=0}^s \sigma_l \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{y}^{(k+1)}(t_n) l^k h^k \\ &= \left( \sum_{l=0}^s \rho_l \right) \mathbf{y}(t_n) + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{l=0}^s l^k \rho_l - k \sum_{l=0}^s l^{k-1} \sigma_l \right) h^k \mathbf{y}^{(k)}(t_n). \end{aligned}$$

Thus, to obtain  $\mathcal{O}(h^{p+1})$  regardless of the choice of  $\mathbf{y}$ , it is necessary and sufficient that

$$\sum_{l=0}^s \rho_l = 0, \quad \sum_{l=0}^s l^k \rho_l = k \sum_{l=0}^s l^{k-1} \sigma_l, \quad k = 1, 2, \dots, p. \quad (4.7)$$

On the other hand, expanding again into Taylor series,

$$\begin{aligned} \rho(e^z) - z\sigma(e^z) &= \sum_{l=0}^s \rho_l e^{lz} - z \sum_{l=0}^s \sigma_l e^{lz} = \sum_{l=0}^s \rho_l \left( \sum_{k=0}^{\infty} \frac{1}{k!} l^k z^k \right) - z \sum_{l=0}^s \sigma_l \left( \sum_{k=0}^{\infty} \frac{1}{k!} l^k z^k \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{l=0}^s l^k \rho_l \right) z^k - \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left( \sum_{l=0}^s l^{k-1} \sigma_l \right) z^k \\ &= \left( \sum_{l=0}^s \rho_l \right) + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \sum_{l=0}^s l^k \rho_l - k \sum_{l=0}^s l^{k-1} \sigma_l \right) z^k. \end{aligned}$$

The theorem follows from (4.7). □

**Example** For the 2-step *Adams–Bashforth method* we have  $\rho(w) = w^2 - w$ ,  $\sigma(w) = \frac{3}{2}w - \frac{1}{2}$  and so

$$\rho(e^z) - z\sigma(e^z) = [1 + 2z + 2z^2 + \frac{4}{3}z^3] - [1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3] - \frac{3}{2}z[1 + z + \frac{1}{2}z^2] + \frac{1}{2}z + \mathcal{O}(z^4) = \frac{5}{12}z^3 + \mathcal{O}(z^4).$$

Hence the method is of order 2.