Mathematical Tripos Part IB: Lent 2019

Numerical Analysis – Lecture 3[1]

2 Orthogonal polynomials

2.1 Orthogonality in general linear spaces

We have already seen the scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} x_i y_i$, acting on $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Likewise, given arbitrary *weights* $w_1, w_2, \ldots, w_n > 0$, we may define $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} w_i x_i y_i$. In general, a *scalar* (or *inner*) *product* is any function $\mathbb{V} \times \mathbb{V} \to \mathbb{R}$, where $\mathbb{V}$ is a vector space over the reals, subject to the following three axioms:
Symmetry: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$;
Nonnegativity: $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0 \ \forall \boldsymbol{x} \in \mathbb{V}$ and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$; and
Linearity: $\langle a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z} \rangle = a\langle \boldsymbol{x}, \boldsymbol{z} \rangle + b\langle \boldsymbol{y}, \boldsymbol{z} \rangle \ \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{V}$, $a, b \in \mathbb{R}$.
Given a scalar product, we may define *orthogonality:* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$ are orthogonal if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.
Let $\mathbb{V} = C[a, b]$, $w \in \mathbb{V}$ be a fixed *positive* function and define

$$\langle f, g \rangle := \int_a^b w(x) f(x) g(x) \, \mathrm{d}x \tag{2.1}$$

for all $f, g \in \mathbb{V}$. It is easy to verify all three axioms of the scalar product.

2.2 Orthogonal polynomials – definition, existence, uniqueness

Given a scalar product in $\mathbb{V} = \mathbb{P}[x]$ (the vector space of polynomials in x, with no bound on the degree), we seek to define a sequence of polynomials $p_0, p_1, p_2, \ldots$ such that:

- $\deg(p_n) = n$ for all $n \geq 0$; and
- $\langle p_n, p_m \rangle = 0$ for all $n \neq m$.

This sequence will be called the *orthogonal polynomials*, and p_n will be called the *n'th orthogonal polynomial*. Observe that for such sequence, $(p_0, \ldots, p_n)$ is an orthogonal basis of $\mathbb{P}_n[x]$ for any $n \geq 0$. **Note**: Different scalar products in general lead to different orthogonal polynomials.

The existence of orthogonal polynomials is the object of the next theorem. A polynomial in $\mathbb{P}_n[x]$ is *monic* if the coefficient of x^n therein equals one.

Theorem For every $n \geq 0$ there exists a unique monic orthogonal polynomial p_n of degree n.

Proof. We let $p_0(x) \equiv 1$ and prove the theorem by induction on n. Thus, suppose that $p_0, p_1, \ldots, p_n$ satisfy the induction hypothesis. To define p_{n+1} let $q(x) := x^{n+1} \in \mathbb{P}_{n+1}[x]$ and, motivated by the *Gram–Schmidt algorithm,* choose

$$p_{n+1}(x) = q(x) - \sum_{k=0}^{n} \frac{\langle q, p_k \rangle}{\langle p_k, p_k \rangle} p_k(x). \tag{2.2}$$

Clearly, $p_{n+1} \in \mathbb{P}_{n+1}[x]$ and it is monic (since all the terms in the sum are of degree $\leq n$).
Let $m \in \{0, 1, \ldots, n\}$. It follows from (2.2) and the induction hypothesis that

$$\langle p_{n+1}, p_m \rangle = \langle q, p_m \rangle - \sum_{k=0}^{n} \frac{\langle q, p_k \rangle}{\langle p_k, p_k \rangle} \langle p_k, p_m \rangle = \langle q, p_m \rangle - \frac{\langle q, p_m \rangle}{\langle p_m, p_m \rangle} \langle p_m, p_m \rangle = 0.$$

Hence, p_{n+1} is orthogonal to $p_0, \ldots, p_n$. To prove uniqueness, we suppose the existence of two monic orthogonal polynomials $p_{n+1}, \tilde{p}_{n+1} \in \mathbb{P}_{n+1}[x]$. Let $p := p_{n+1} - \tilde{p}_{n+1} \in \mathbb{P}_n[x]$, hence $\langle p_{n+1}, p \rangle = \langle \tilde{p}_{n+1}, p \rangle = 0$, and this implies

$$0 = \langle p_{n+1}, p \rangle - \langle \tilde{p}_{n+1}, p \rangle = \langle p_{n+1} - \tilde{p}_{n+1}, p \rangle = \langle p, p \rangle,$$

and we deduce $p \equiv 0$. $\square$

[1]Corrections and suggestions to these notes should be emailed to `h.fawzi@damtp.cam.ac.uk`.

Example *Legendre polynomials* Define the scalar product $\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$ for $f, g \in \mathbb{P}[x]$. The orthogonal polynomials arising from this scalar product is called *Legendre polynomials.* The first polynomials of this sequence are:

$$p_0(x) = 1,\ \ p_1(x) = x,\ \ p_2(x) = x^2 - (1/3),\ \ p_3(x) = x^3 - (3/5)x,\ \ p_4(x) = x^4 - (30/35)x^2 + (3/35).$$

Well-known examples of orthogonal polynomials include:

Name	Notation	Interval $[a, b]$	Weight function
Legendre	P_n	$[-1, 1]$	$w(x) \equiv 1$
Chebyshev	T_n	$[-1, 1]$	$w(x) = (1 - x^2)^{-1/2}$
Laguerre	L_n	$[0, \infty)$	$w(x) = \mathrm{e}^{-x}$
Hermite	H_n	$(-\infty, \infty)$	$w(x) = \mathrm{e}^{-x^2}$

The weight function refers to the function w in the scalar product definition of Equation (2.1).

2.3 The three-term recurrence relation

How to construct orthogonal polynomials? (2.2) might help, but it suffers from loss of accuracy due to imprecisions in the calculation of scalar products. A considerably better procedure follows from our next theorem. For the next theorem we assume the scalar product satisfies $\langle xp, q\rangle = \langle p, xq\rangle$ for any $p, q \in \mathbb{P}[x]$.

Theorem Assuming the scalar product on $\mathbb{P}[x]$ satisfies $\langle xp, q\rangle = \langle p, xq\rangle$ for all $p, q \in \mathbb{P}[x]$, monic orthogonal polynomials are given by the formula

$$\begin{aligned} &p_{-1}(x) \equiv 0, \qquad\qquad p_0(x) \equiv 1, \\ &p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad n = 0, 1, \ldots, \end{aligned} \tag{2.3}$$

where

$$\alpha_n := \frac{\langle p_n, xp_n\rangle}{\langle p_n, p_n\rangle}, \qquad \beta_n = \frac{\langle p_n, p_n\rangle}{\langle p_{n-1}, p_{n-1}\rangle} > 0.$$

Remark: The assumption $\langle xp, q\rangle = \langle p, xq\rangle$ on the scalar product is satisfied by most common examples of scalar products. It is satisfied for example by (2.1).

Proof. Pick $n \geq 0$ and let $\psi(x) := p_{n+1}(x) - (x - \alpha_n)p_n(x) + \beta_n p_{n-1}(x)$. Since p_n and p_{n+1} are monic, it follows that $\psi \in \mathbb{P}_n[x]$. Moreover, because of orthogonality of p_{n-1}, p_n, p_{n+1},

$$\langle \psi, p_\ell\rangle = \langle p_{n+1}, p_\ell\rangle - \langle p_n, (x - \alpha_n)p_\ell\rangle + \beta_n\langle p_{n-1}, p_\ell\rangle = 0, \qquad \ell = 0, 1, \ldots, n-2.$$

Because of monicity, $xp_{n-1} = p_n + q$, where $q \in \mathbb{P}_{n-1}[x]$. Thus, from the definition of α_n, β_n,

$$\begin{aligned} \langle \psi, p_{n-1}\rangle &= -\langle p_n, xp_{n-1}\rangle + \beta_n\langle p_{n-1}, p_{n-1}\rangle = -\langle p_n, p_n\rangle + \beta_n\langle p_{n-1}, p_{n-1}\rangle = 0, \\ \langle \psi, p_n\rangle &= -\langle xp_n, p_n\rangle + \alpha_n\langle p_n, p_n\rangle = 0. \end{aligned}$$

Every $p \in \mathbb{P}_n[x]$ that obeys $\langle p, p_\ell\rangle = 0$, $\ell = 0, 1, \ldots, n$, must necessarily be the zero polynomial. For suppose that it is not so and let x^s be the highest power of x in p. Then $\langle p, p_s\rangle \neq 0$, which is impossible. We deduce that $\psi \equiv 0$, hence (2.3) is true. □

Example *Chebyshev polynomials* We choose the scalar product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\frac{\mathrm{d}x}{\sqrt{1 - x^2}}, \qquad f, g \in C[-1, 1]$$

and define $T_n \in \mathbb{P}_n[x]$ by the relation $T_n(\cos\theta) = \cos(n\theta)$. Hence $T_0(x) \equiv 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$ etc. Changing the integration variable,

$$\langle T_n, T_m\rangle = \int_{-1}^{1} T_n(x)T_m(x)\frac{\mathrm{d}x}{\sqrt{1 - x^2}} = \int_0^\pi \cos n\theta \cos m\theta\,\mathrm{d}\theta = \tfrac{1}{2}\int_0^\pi [\cos(n{+}m)\theta + \cos(n{-}m)\theta]\,\mathrm{d}\theta = 0$$

whenever $n \neq m$. The recurrence relation for Chebyshev polynomials is particularly simple, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, as can be verified at once from the identity $\cos[(n+1)\theta] + \cos[(n-1)\theta] = 2\cos(\theta)\cos(n\theta)$. Note that the T_ns aren't monic, hence the inconsistency with (2.3). To obtain monic polynomials take $T_n(x)/2^{n-1}$, $n \geq 1$.