Mathematical Tripos Part IB: Lent 2020

Numerical Analysis – Lecture 6[1]

We look at another example of application of the Peano kernel theorem.

Example We approximate a derivative by a linear combination of function values, $f'(0) \approx -\frac{3}{2}f(0)+2f(1)-\frac{1}{2}f(2)$. Define $L(f) := f'(0) - [-\frac{3}{2}f(0) + 2f(1) - \frac{1}{2}f(2)]$ and it is easy to check that $L(f) = 0$ for $f \in \mathbb{P}_2[x]$. (Verify by trying $f(x) = 1, x, x^2$ and using linearity of L.) Thus, for $f \in \mathrm{C}^3[0,2]$ we have

$$L(f) = \tfrac{1}{2}\int_0^2 K(\theta) f'''(\theta)\,\mathrm{d}\theta$$

with $K(\theta) = L(x \mapsto (x-\theta)_+^2)$. For fixed θ, let $g(x) := (x-\theta)_+^2$. Then

$$\begin{aligned} K(\theta) = L(g) &= g'(0) - \left[-\tfrac{3}{2}g(0) + 2g(1) - \tfrac{1}{2}g(2)\right] \\ &= 2(0-\theta)_+ - \left[-\tfrac{3}{2}(0-\theta)_+^2 + 2(1-\theta)_+^2 - \tfrac{1}{2}(2-\theta)_+^2\right] \\ &= \begin{cases} 2\theta - \frac{3}{2}\theta^2, & 0 \le \theta \le 1, \\ \frac{1}{2}(2-\theta)^2, & 1 \le \theta \le 2, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

One can verify that $\int_0^2 |K(\theta)|\,\mathrm{d}\theta = \frac{2}{3}$. Consequently for any $f \in \mathrm{C}^3[0,2]$ we have

$$|L(f)| \le \frac{1}{2!}\int_0^2 |K(\theta) f'''(\theta)|\,\mathrm{d}\theta \le \frac{1}{2}\|f'''\|_\infty \int_0^2 |K(\theta)|\,\mathrm{d}\theta = \frac{1}{3}\|f'''\|_\infty,$$

where $\|f'''\|_\infty = \max_{x\in[0,2]} |f'''(x)|$.

4 Ordinary differential equations

We wish to approximate the exact solution of the *ordinary differential equation (ODE)*

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \qquad t \ge 0, \tag{4.1}$$

where $\boldsymbol{y} \in \mathbb{R}^N$ and the function $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is sufficiently 'nice'. (In principle, it is enough for $\boldsymbol{f}$ to be Lipschitz to ensure that the solution exists and is unique. Yet, for simplicity, we henceforth assume that $\boldsymbol{f}$ is analytic: in other words, we are always able to expand locally into Taylor series.) The equation (4.1) is accompanied by the initial condition $\boldsymbol{y}(0) = \boldsymbol{y}_0$.
Our purpose is to approximate $\boldsymbol{y}_{n+1} \approx \boldsymbol{y}(t_{n+1})$, $n = 0, 1, \ldots$, where $t_m = mh$ and the *time step* $h > 0$ is small, from $\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ and equation (4.1).

4.1 One-step methods

A one-step method is a map $\boldsymbol{y}_{n+1} = \boldsymbol{\varphi}_h(t_n, \boldsymbol{y}_n)$, i.e. an algorithm which allows $\boldsymbol{y}_{n+1}$ to depend only on t_n, $\boldsymbol{y}_n$, h and the ODE (4.1).

The Euler method: We know $\boldsymbol{y}$ and its slope $\boldsymbol{y}'$ at $t = 0$ and wish to approximate $\boldsymbol{y}$ at $t = h > 0$. The most obvious approach is to truncate $\boldsymbol{y}(h) = \boldsymbol{y}(0) + h\boldsymbol{y}'(0) + \frac{1}{2}h^2\boldsymbol{y}''(0) + \cdots$ at the h^2 term. Since $\boldsymbol{y}'(0) = \boldsymbol{f}(t_0, \boldsymbol{y}_0)$, this procedure approximates $\boldsymbol{y}(h) \approx \boldsymbol{y}_0 + h\boldsymbol{f}(t_0, \boldsymbol{y}_0)$ and we thus set $\boldsymbol{y}_1 = \boldsymbol{y}_0 + h\boldsymbol{f}(t_0, \boldsymbol{y}_0)$. By the same token, we may advance from h to $2h$ by letting $\boldsymbol{y}_2 = \boldsymbol{y}_1 + h\boldsymbol{f}(t_1, \boldsymbol{y}_1)$. In general, we obtain the *Euler method*

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n), \qquad n = 0, 1, \ldots. \tag{4.2}$$

[1] Corrections and suggestions to these notes should be emailed to `h.fawzi@damtp.cam.ac.uk`.

Convergence: Let $t^* > 0$ be given. We say that a method, which for every $h > 0$ produces the solution sequence $\boldsymbol{y}_n = \boldsymbol{y}_n(h)$, $n = 0, 1, \ldots, \lfloor t^*/h \rfloor$, *converges* if

$$\lim_{h \to 0} \max_{n=0,\ldots,\lfloor t^*/h \rfloor} \|\boldsymbol{y}_n(h) - \boldsymbol{y}(nh)\| = 0,$$

where $\boldsymbol{y}(nh)$ is the evaluation at time $t = nh$ of the exact solution of (4.1).

Theorem *Suppose that* $\boldsymbol{f}$ *satisfies the Lipschitz condition: there exists* $\lambda \geq 0$ *such that*

$$\|\boldsymbol{f}(t, \boldsymbol{v}) - \boldsymbol{f}(t, \boldsymbol{w})\| \leq \lambda \|\boldsymbol{v} - \boldsymbol{w}\|, \qquad t \in [0, t^*], \quad \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^N.$$

Then the Euler method (4.2) converges.

Proof. Let $\boldsymbol{e}_n = \boldsymbol{y}_n - \boldsymbol{y}(t_n)$, the error at step n, where $0 \leq n \leq t^*/h$. Thus,

$$\boldsymbol{e}_{n+1} = \boldsymbol{y}_{n+1} - \boldsymbol{y}(t_{n+1}) = [\boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)] - [\boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n) + \mathcal{O}(h^2)].$$

By the Taylor theorem, the $\mathcal{O}(h^2)$ term can be bounded uniformly for all $[0, t^*]$ in the underlying norm $\|\cdot\|$ by ch^2, where $c > 0$ (Indeed if we take $c = \frac{1}{2} \max_{t \in [0,t^*]} \|\boldsymbol{y}''(t)\|$, then by Taylor's formula with integral remainder we get that for any t, h such that $0 \leq t < t + h \leq t^*$, $\|\boldsymbol{y}(t+h) - (\boldsymbol{y}(t) + h\boldsymbol{y}'(t))\| \leq ch^2$.) Thus, using (4.1) and the triangle inequality,

$$\begin{aligned} \|\boldsymbol{e}_{n+1}\| &\leq \|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| + h\|\boldsymbol{f}(t_n, \boldsymbol{y}_n) - \boldsymbol{f}(t_n, \boldsymbol{y}(t_n))\| + ch^2 \\ &\leq \|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| + h\lambda\|\boldsymbol{y}_n - \boldsymbol{y}(t_n)\| + ch^2 = (1 + h\lambda)\|\boldsymbol{e}_n\| + ch^2. \end{aligned}$$

Consequently, by induction,

$$\|\boldsymbol{e}_{n+1}\| \leq (1 + h\lambda)^m \|\boldsymbol{e}_{n+1-m}\| + ch^2 \sum_{j=0}^{m-1} (1 + h\lambda)^j, \qquad m = 0, 1, \ldots, n+1.$$

In particular, letting $m = n + 1$ and bearing in mind that $\boldsymbol{e}_0 = \boldsymbol{0}$, we have

$$\|\boldsymbol{e}_{n+1}\| \leq ch^2 \sum_{j=0}^{n} (1 + h\lambda)^j = ch^2 \frac{(1 + h\lambda)^{n+1} - 1}{(1 + h\lambda) - 1} \leq \frac{ch}{\lambda} (1 + h\lambda)^{n+1}.$$

For small $h > 0$ it is true that $0 < 1 + h\lambda \leq \mathrm{e}^{h\lambda}$. This and $(n+1)h \leq t^*$ imply that $(1 + h\lambda)^{n+1} \leq \mathrm{e}^{t^*\lambda}$, therefore $\|\boldsymbol{e}_n\| \leq \frac{c\mathrm{e}^{t^*\lambda}}{\lambda} h \xrightarrow{h \to 0} 0$ uniformly for $0 \leq nh \leq t^*$ and the theorem is true. $\square$