Mathematical Tripos Part IB: Lent 2020

Numerical Analysis – Lecture 10[1]

Example We have already seen that the linear stability domain of the explicit Euler's method is $\mathcal{D} = \{z \in \mathbb{C} : |1+z| < 1\}$ (not A-stable), and for the implicit Euler's method it is $\mathcal{D} = \{z \in \mathbb{C} : |1-z| > 1\}$ (A-stable). Consider now the trapezoidal rule: $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h[\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})]$. Applied to $y' = \lambda y$ we get $y_{n+1} = [(1+\frac{1}{2}h\lambda)/(1-\frac{1}{2}h\lambda)]y_n$ thus, by induction, $y_n = [(1+\frac{1}{2}h\lambda)/(1-\frac{1}{2}h\lambda)]^n y_0$. Therefore

$$z \in \mathcal{D} \qquad \Leftrightarrow \qquad \left|\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}\right| < 1 \qquad \Leftrightarrow \qquad \operatorname{Re} z < 0$$

and we deduce that $\mathcal{D} = \mathbb{C}^-$. Hence, the method is A-stable.

Discussion A-stability analysis of multistep methods is considerably more complicated. However, according to the *second Dahlquist barrier,* no multistep method of order $p \geq 3$ may be A-stable. Note that the $p = 2$ barrier for A-stability is attained by the trapezoidal rule.

The Dahlquist barrier implies that, in our quest for higher-order methods with good stability properties, we need to pursue one of the following strategies:

- either relax the definition of A-stability
- or consider other methods in place of multistep.

The two courses of action will be considered next.

Stiffness and BDF methods Inasmuch as no multistep method of order $p \geq 3$ may be A-stable, stability properties of BDF, say, are satisfactory for most stiff equations. The point is that in many stiff linear systems in applications the eigenvalues are not just in $\mathbb{C}^-$ but also well away from $\mathrm{i}\mathbb{R}$. *[Analysis of nonlinear stiff equations is difficult and well outside the scope of this course.]* All BDF methods of order $p \leq 6$ (i.e., all convergent BDF methods) share the feature that the linear stability domain $\mathcal{D}$ includes a wedge about $(-\infty, 0)$: such methods are said to be A_0*-stable.*

Stiffness and Runge–Kutta Unlike multistep methods, implicit high-order RK may be A-stable. For example, consider the following 2-stage implicit RK method:

$$\begin{aligned}
\boldsymbol{k}_1 &= \boldsymbol{f}\left(t_n, \boldsymbol{y}_n + \tfrac{1}{4}h(\boldsymbol{k}_1 - \boldsymbol{k}_2)\right), \\
\boldsymbol{k}_2 &= \boldsymbol{f}\left(t_n + \tfrac{2}{3}h, \boldsymbol{y}_n + \tfrac{1}{12}h(3\boldsymbol{k}_1 + 5\boldsymbol{k}_2)\right), \\
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \tfrac{1}{4}h(\boldsymbol{k}_1 + 3\boldsymbol{k}_2).
\end{aligned}$$

One can show that this method is A-stable and has order 3. We first show that it is A-stable. Applying the method to $y' = \lambda y$, we have

$$\begin{aligned}
hk_1 &= h\lambda\left(y_n + \tfrac{1}{4}hk_1 - \tfrac{1}{4}hk_2\right), \\
hk_2 &= h\lambda\left(y_n + \tfrac{1}{4}hk_1 + \tfrac{5}{12}hk_2\right).
\end{aligned}$$

This is a linear system, whose solution is

$$\begin{bmatrix} hk_1 \\ hk_2 \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{4}h\lambda & \frac{1}{4}h\lambda \\ -\frac{1}{4}h\lambda & 1-\frac{5}{12}h\lambda \end{bmatrix}^{-1} \begin{bmatrix} h\lambda y_n \\ h\lambda y_n \end{bmatrix} = \frac{h\lambda y_n}{1-\frac{2}{3}h\lambda+\frac{1}{6}(h\lambda)^2} \begin{bmatrix} 1-\frac{2}{3}h\lambda \\ 1 \end{bmatrix},$$

therefore

$$y_{n+1} = y_n + \tfrac{1}{4}hk_1 + \tfrac{3}{4}hk_2 = \frac{1+\frac{1}{3}h\lambda}{1-\frac{2}{3}h\lambda+\frac{1}{6}h^2\lambda^2} y_n.$$

Let

$$r(z) = \frac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}.$$

[1] Corrections and suggestions to these notes should be emailed to `h.fawzi@damtp.cam.ac.uk`.

Then $y_{n+1} = r(h\lambda)y_n$, therefore, by induction, $y_n = [r(h\lambda)]^n y_0$ and we deduce that

$$\mathcal{D} = \{z \in \mathbb{C} \,:\, |r(z)| < 1\}$$

We wish to prove that $|r(z)| < 1$ for every $z \in \mathbb{C}^-$, since this is equivalent to A-stability. This will be done by a technique that can be applied to other RK methods. According to the *maximum modulus principle* from Complex Methods, if g is a nonconstant analytic function defined on an open set $\Omega \subset \mathbb{C}$, then $|g|$ has no maximum in Ω. We let $g = r$. This is a rational function, hence its only singularities are the poles $2 \pm \mathrm{i}\sqrt{2}$ and g is analytic in $\Omega = \mathbb{C}^- = \{z \in \mathbb{C} \,:\, \mathrm{Re}\, z < 0\}$. Thus to prove that the method is A-stable, it suffices to check that $\lim_{|z|\to\infty, \mathrm{Re}\,[z]<0} |r(z)| \le 1$ and that $|r(\mathrm{i}t)| \le 1$ for all $t \in \mathbb{R}$. The first condition is easy to check from the definition of r. For the second condition, we verify that

$$|r(\mathrm{i}t)|^2 \le 1 \qquad \Leftrightarrow \qquad |1 - \tfrac{2}{3}\mathrm{i}t - \tfrac{1}{6}t^2|^2 - |1 + \tfrac{1}{3}\mathrm{i}t|^2 \ge 0.$$

But $|1 - \frac{2}{3}\mathrm{i}t - \frac{1}{6}t^2|^2 - |1 + \frac{1}{3}\mathrm{i}t|^2 = \frac{1}{36}t^4 \ge 0$ and it follows that the method is A-stable.

Let us now show that the method has order 3. To do this we restrict our attention to scalar, autonomous equations of the form $y' = f(y)$. For brevity, we use the convention that all functions are evaluated at $y = y(t_n)$, e.g. $f_y = \mathrm{d}f(y(t_n))/\mathrm{d}y$. Thus,

$$\begin{aligned} k_1 &= f + \tfrac{1}{4}hf_y(k_1 - k_2) + \tfrac{1}{32}h^2 f_{yy}(k_1 - k_2)^2 + \mathcal{O}\big(h^3\big)\,, \\ k_2 &= f + \tfrac{1}{12}hf_y(3k_1 + 5k_2) + \tfrac{1}{288}h^2 f_{yy}(3k_1 + 5k_2)^2 + \mathcal{O}\big(h^3\big)\,. \end{aligned}$$

We have $k_1, k_2 = f + \mathcal{O}(h)$ and substitution in the above equations yields $k_1 = f + \mathcal{O}\big(h^2\big)$, $k_2 = f + \frac{2}{3}hf_y f + \mathcal{O}\big(h^2\big)$. Substituting again, we obtain

$$\begin{aligned} & k_1 = f - \tfrac{1}{6}h^2 f_y^2 f + \mathcal{O}\big(h^3\big)\,, \\ & k_2 = f + \tfrac{2}{3}hf_y f + h^2\left(\tfrac{5}{18}f_y^2 f + \tfrac{2}{9}f_{yy}f^2\right) + \mathcal{O}\big(h^3\big) \\ \Rightarrow \quad & y_{n+1} = y + hf + \tfrac{1}{2}h^2 f_y f + \tfrac{1}{6}h^3(f_y^2 f + f_{yy}f^2) + \mathcal{O}\big(h^4\big)\,. \end{aligned}$$

But $y' = f \Rightarrow y'' = f_y f \Rightarrow y''' = f_y^2 f + f_{yy}f^2$ and we deduce from Taylor's theorem that the method is at least of order 3. (It is easy to verify that it isn't of order 4, for example applying it to the equation $y' = \lambda y$.)

Example It is possible to prove that the 2-stage *Gauss–Legendre method*

$$\begin{aligned} \boldsymbol{k}_1 &= \boldsymbol{f}(t_n + (\tfrac{1}{2} - \tfrac{\sqrt{3}}{6})h, \boldsymbol{y}_n + \tfrac{1}{4}h\boldsymbol{k}_1 + (\tfrac{1}{4} - \tfrac{\sqrt{3}}{6})h\boldsymbol{k}_2), \\ \boldsymbol{k}_2 &= \boldsymbol{f}(t_n + (\tfrac{1}{2} + \tfrac{\sqrt{3}}{6})h, \boldsymbol{y}_n + (\tfrac{1}{4} + \tfrac{\sqrt{3}}{6})h\boldsymbol{k}_1 + \tfrac{1}{4}h\boldsymbol{k}_2), \\ \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \tfrac{1}{2}h(\boldsymbol{k}_1 + \boldsymbol{k}_2) \end{aligned}$$

is of order 4. [You can do this for $y' = f(y)$ by expansion, but it becomes messy for $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$.] It can be easily verified that for $y' = \lambda y$ we have $y_n = [r(h\lambda)]^n y_0$, where $r(z) = (1 + \frac{1}{2}z + \frac{1}{12}z^2)/(1 - \frac{1}{2}z + \frac{1}{12}z^2)$. Since the poles of r reside at $3 \pm \mathrm{i}\sqrt{3}$ and $|r(\mathrm{i}t)| \equiv 1$, we can again use the maximum modulus principle to argue that $\mathcal{D} = \mathbb{C}^-$ and the Gauss–Legendre method is A-stable.