1 Introduction

In this course we are interested in solving optimization problems:

$$\min \quad f(x) \quad \text{subject to} \quad x \in X$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the *objective (or cost) function* and $X \subseteq \mathbb{R}^n$ is the *feasible set*. Optimization problems show up in many areas:

Applications of optimization

- Least-squares/classification: Given data points $(x_1, y_1), \ldots, (x_n, y_n)$ where $x_i \in \mathbb{R}^p$ and $y_i \in \mathbb{R}$ we want to find $w \in \mathbb{R}^p$ and $b \in \mathbb{R}$ such that $y_i \approx w^T x_i + b$. A common way to find such a w, b is to solve
$$\min_{w\in\mathbb{R}^p, b\in\mathbb{R}} \quad \sum_{i=1}^n (w^T x_i + b - y_i)^2. \tag{1}$$
Having solved this optimization problem and obtained the optimal w, b, the predicted output $\bar{y}$ for a new data point $\bar{x}$ is $\bar{y} = w^T\bar{x} + b$.
If $y_i \in \{-1, +1\}$ (classification problem), it is more common to use a logistic loss rather than a least-squares loss. This leads to the optimization problem
$$\min_{w\in\mathbb{R}^p, b\in\mathbb{R}} \quad \sum_{i=1}^n \log_2\left(1 + e^{-y_i(w^T x_i + b)}\right). \tag{2}$$
Having solved this optimization problem and obtained the optimal w, b, the predicted class $\bar{y}$ for a new data point $\bar{x}$ is $\bar{y} = \operatorname{sign}(w^T\bar{x} + b)$. In nonlinear classification, we have a family of functions $F = \{f_w : w \in \mathbb{R}^p\}$ indexed by some real vector $w \in \mathbb{R}^p$. For example f_w could be a neural network with weight vector w. The training problem, with a logistic loss, then becomes
$$\min_{w\in\mathbb{R}^p, b\in\mathbb{R}} \quad \sum_{i=1}^n \log_2\left(1 + e^{-y_i f_w(x)}\right).$$

- Geometry: given a cloud of point $x_1, \ldots, x_n \in \mathbb{R}^p$, we want to find the ellipsoid E of minimum volume that contains the points, i.e., we want to solve
$$\min \quad \text{volume}(E) \quad \text{s.t.} \quad x_i \in E \quad \forall i = 1, \ldots, n.$$
Assuming (for simplicity) that the ellipsoid is centered at the origin, we can write $E = \left\{z \in \mathbb{R}^p : z^T Q^{-1} z \le 1\right\}$ where Q is a $p \times p$ real symmetric matrix that is positive definite. Then the volume of E is proportional to $\det(Q)$. Thus our problem can be written as
$$\min \quad \det(Q) \quad \text{s.t.} \quad \begin{cases} Q \text{ is positive definite} \\ x_i^T Q^{-1} x_i \le 1. \end{cases} \tag{3}$$

- Graph theory: given a graph $G = (V, E)$ where $E \subset \binom{V}{2}$, a *stable set* of G is a subset S of vertices that are pairwise nonadjacent, i.e., $i, j \in S \Rightarrow \{i, j\} \notin E$. The maximum stable set problem asks for the largest stable set in a given graph G

  $$\max \quad |S| \quad \text{s.t.} \quad S \text{ stable set.}$$

  Such a problem can be reformulated as a constrained optimization over $\mathbb{R}^n$ by considering the characteristic vector x of S:

  $$\max_{x \in \mathbb{R}^n} \quad \sum_{i=1}^n x_i \quad \text{s.t.} \quad \begin{cases} x_i^2 = x_i & \forall i = 1, \dots, n \\ x_i x_j = 0 & \forall \{i, j\} \in E. \end{cases}$$

Optimization on the cube To illustrate some of the concepts in this course consider the problem of minimizing a function $f : \mathbb{R}^n \to \mathbb{R}$ on $[0, 1]^n$, i.e., to compute:

$$f^* = \min_{x \in [0,1]^n} \quad f(x).$$

Our goal will be to find a solution with accuracy $\epsilon > 0$:

$$\text{Find } \bar{x} \text{ s.t. } f(\bar{x}) - f^* \leq \epsilon. \tag{*}$$

The algorithms have access to f through a *black box* which, given an input $x \in [0, 1]^n$ returns the value $f(x) \in \mathbb{R}$. This is called an zeroth-order oracle model[1] The complexity of an algorithm on a given function f is the number of queries it makes to the oracle. So a general algorithm has the following form:

1. Query oracle at $x_0 \in [0, 1]^n$ to get value $f_0 = f(x_0)$
2. Query oracle at $x_1 \in [0, 1]^n$ (allowed to depend on f_0) to get value $f_1 = f(x_1)$
3. Query oracle at $x_2 \in [0, 1]^n$ (allowed to depend on f_0, f_1) to get value $f_2 = f(x_2)$
4. ...
5. Query oracle at $x_{N-1} \in [0, 1]^n$ (allowed to depend on $f_0, \dots, f_{N-2}$) to get value $f_{N-1} = f(x_{N-1})$
6. Output $\bar{x}$ based on the gathered information about f

We will consider the class of functions that are L-Lipschitz with respect to ℓ_∞ norm

$$\mathcal{F}_L = \{f : [0, 1]^n \to \mathbb{R} \text{ s.t. } |f(x) - f(y)| \leq L\|x - y\|_\infty \ \forall x, y \in [0, 1]^n\}$$

where $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$. We can prove the following:

Proposition 1.1. *There is an algorithm that can return an ϵ-accurate minimizer (in the sense of (*)) of any $f \in \mathcal{F}_L$ with a number of queries $\leq (\lfloor \frac{L}{2\epsilon} \rfloor + 2)^n$.*

[1] A first-order oracle returns the gradient of f at x, and a second-order oracle returns the Hessian of f at x. We will see this later...

Proof. Grid search. We discretize the cube $[0,1]^n$ using grid points that are equispaced by $2\epsilon/L$ in each dimension. Let $(x_i)_{i=1,\dots,N}$ be the grid points; there are $N \le (\lfloor \frac{L}{2\epsilon} \rfloor + 2)^n$ such grid points (we include points at coordinate 0 and coordinate 1, hence the +2). Let $\bar{x}$ be the grid point where the value of f is smallest, i.e.,

$$\bar{x} = \underset{x \in \{x_1,\dots,x_N\}}{\operatorname{argmin}} f(x).$$

We claim that this algorithm achieves the desired accuracy. Indeed, let x^* be a minimizer of f on $[0,1]^n$, and let $\tilde{x}$ be the closest grid point to x^* in the ℓ_∞ norm. Since the grid is equispaced by $2\epsilon/L$ it is not difficult to see that $\|x^* - \tilde{x}\|_\infty \le \epsilon/L$. Then we have

$$f(\bar{x}) - f^* \le f(\tilde{x}) - f^* \le L\|\tilde{x} - x^*\|_\infty \le \epsilon$$

as desired. □

The algorithm produced in the previous proposition is not great. For functions of large number of variables n the algorithm is not at all practical. Can we do better? The answer turns out to be no, if we want our algorithm to work for all $f \in \mathcal{F}_L$.

Proposition 1.2. *Assume $\mathcal{A}$ is an algorithm that returns an ϵ-accurate minimizer for all $f \in \mathcal{F}_L$. Then there is at least one function $f \in \mathcal{F}_L$ on which $\mathcal{A}$ does at least $\ge (\lfloor \frac{L}{3\epsilon} \rfloor)^n - 1$ queries.*

Proof. Recall that an algorithm $\mathcal{A}$ is given by a sequence of query points $x_0, x_1, \dots$ where each query point is allowed to depend on the answer received on the previous ones. We are going to simulate the algorithm on the function $f(x) \equiv 0$ (the function equal to zero everywhere). On such a function the algorithm will query certain (fixed) points $x_0, x_1, x_2, \dots, x_{N-1}$ all in $[0,1]^n$ before producing a point $\bar{x} \in [0,1]^n$. Let $S = \{x_0, \dots, x_{N-1}, \bar{x}\}$. We claim that necessarily $|S| \ge (\lfloor L/(3\epsilon) \rfloor)^n$. Fix $\eta = 3\epsilon/L$ and consider dividing $[0,1]^n$ into small boxes each of size η. We have at least $\lfloor 1/\eta \rfloor^n$ disjoint such boxes. Assuming for contradiction that $|S| < (\lfloor 1/\eta \rfloor)^n$, by the pigeonhole principle, there exists at least one box which does not contain any point from S. Let x^* be the center of that box and define the function

$$f(x) = \min(0, L\|x - x^*\|_\infty - \eta L/2).$$

Note that $f \in \mathcal{F}_L$, it is zero outside the box centered at x^* and its minimum is $-\eta L/2 = -3\epsilon/2$. If we run the algorithm on this function f we will get the same output as for the function that is identically zero (the $\bar{x} \in S$ from above). But this $\bar{x}$ is outside the box centered at x^* and so $f(\bar{x}) = 0$. This contradicts the assumption that the algorithm achieves ϵ accuracy on all functions in $\mathcal{F}_L$ because $f(\bar{x}) - f^* = 3\epsilon/2 > \epsilon$. Thus it must be that $|S| \ge \lfloor 1/\eta \rfloor^n = (\lfloor \frac{L}{3\epsilon} \rfloor)^n$. □

We have thus shown that the following min-max quantity

$$\min_{\substack{\text{Algorithms } \mathcal{A} \text{ that achieve} \\ (*) \text{ for all functions in } \mathcal{F}_L}} \max_{f \in \mathcal{F}_L} \text{ Complexity of } \mathcal{A} \text{ on } f$$

lies between $(\frac{L}{3\epsilon})^n$ and $(\frac{L}{2\epsilon} + 2)^n$.

Convex optimization The focus of this course will be on *convex optimization* problems. A convex optimization problem has the form

$$\min_{x \in \mathbb{R}^n} f(x) \text{ subject to } x \in X$$

where $X \subset \mathbb{R}^n$ is a convex set (i.e., $x, y \in X$ and $\lambda \in [0,1]$ implies $\lambda x + (1-\lambda)y \in X$) and where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, i.e.,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Examples of convex optimization problems include:

- Least-squares: $\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
- Linear programming: minimizing a linear function subject to linear inequality constraints
$$\min_{x \in \mathbb{R}^n} \; c^T x \; \text{ s.t. } \; a_1^T x \leq b_1, \ldots, a_m^T x \leq b_m.$$
- Linear classification with logistic loss, see (2)
- Lasso problem (statistics): $\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 + \lambda \|x\|_1$
- The minimum volume enclosing ellipsoid problem in (3) can be formulated as a convex optimization after a suitable change of variables ($P = Q^{-1}$).
- Many others...