Example class 1

1. Prove the separating hyperplane theorem (Theorem 1.1) when C is not necessarily closed.

2. Let C, D be two disjoint convex sets in $\mathbb{R}^n$. Show that there is $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$ such that $\langle a, x\rangle \leq b$ for all $x \in C$ and $\langle a, y\rangle \geq b$ for all $y \in D$. If we assume C and D closed, can we always find a *strict* separating hyperplane?

3. (Carathéodory theorem) Let $S \subset \mathbb{R}^n$. Show that any point in $\text{conv}(S)$ can be written as a convex combination of at most $n+1$ points of S (hint: any k points $s_1, \ldots, s_k$ with $k \geq n+2$ are affinely dependent i.e., there exist $\mu_1, \ldots, \mu_k$ such that $\sum_{i=1}^k \mu_i s_i = 0$ and $\sum_{i=1}^k \mu_i = 0$).

4. Let C be a closed and bounded convex set in $\mathbb{R}^n$ and consider the minimization problem $\min\{\langle c, x\rangle : x \in C\}$ where $c \in \mathbb{R}^n$. Show that there is at least one optimal solution that is an extreme point of C.

5. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ and the inequality $Ax \leq b$ is interpreted componentwise. Show that if x is an extreme point of P then there exists $I \subseteq \{1, \ldots, m\}$ such that $\ker(A_I) = \{0\}$ (where A_I is the matrix obtained from A by keeping only the rows in I), $b \in \text{im}(A_I)$ and $x = A_I^{-1} b_I$. Deduce that P has a finite number of extreme points and give an upper bound for the number of extreme points in terms of m and n. [*Note:* an extreme point of a polyhedron is sometimes called a *basic feasible point*, especially in the context of the simplex algorithm for linear programming. The set I that identifies the extreme point is known as the *basis* of the basic feasible point].

6. Let $P = \text{conv}(v_1, \ldots, v_N)$ where $v_1, \ldots, v_N \in \mathbb{R}^n$. Show that P is the intersection of a finite number of halfspaces. Give an upper bound on the number of halfspaces needed. (*Hint:* Assume (without loss of generality) that P has nonempty interior and $0 \in \text{int}(P)$, and apply the result of the previous question to the polyhedron $P^o = \{y \in \mathbb{R}^n : \langle y, v_i\rangle \leq 1, \ \forall i = 1, \ldots, N\}$; P^o is known as the *polar* of P).

7. (Linear image of a polyhedron) Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and assume that P is bounded. Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that $\pi(P)$ is also a polyhedron, i.e., it is the intersection of a finite number of halfspaces. (*Hint:* Use questions 2 and 3 above. *Note:* the assumption P is bounded is for simplicity; the result is still true if P is not bounded).

8. (Farkas' lemma) Let $K = \{x \in \mathbb{R}^n : Ax \geq 0\}$ be a polyhedral cone. Show that $K^* = \text{cone}(a_1, \ldots, a_m)$ where $a_1, \ldots, a_m$ are the rows of A. Deduce Farkas' lemma (conic version): if $\langle y, x\rangle \geq 0$ for all $x \in K$ then there exists $\lambda \geq 0$ such that $y^T = \lambda^T A$.

9. Let K be a closed convex cone in $\mathbb{R}^n$.

   (a) Show that the following conditions are equivalent:

      (i) K has nonempty interior

      (ii) $\text{span}(K) = \mathbb{R}^n$

      (iii) For any $w \in \mathbb{R}^n \setminus \{0\}$ there exists $x \in K$ such that $\langle w, x\rangle \neq 0$.

   (b) Show that K is pointed if and only K^* has nonempty interior.

(c) Show that $y \in \text{int}(K^*)$ if and only if $\langle y, x\rangle > 0$ for all $x \in K \setminus \{0\}$.

10. In this exercise we will prove Minkowski's theorem for closed convex pointed cones (Theorem 2.2 in lecture 2). Let K be a closed convex cone.

    (a) Assume that there exists $y \in \mathbb{R}^n$ such that $\langle y, x\rangle > 0$ for all $x \in K \setminus \{0\}$. Show how to prove the theorem in this case. *(hint: define $C = \{x \in K \text{ s.t. } \langle y, x\rangle = 1\}$; show that C is a compact convex set and apply Minkowski's theorem for compact convex sets to C).*

    (b) Use Question 9 to show that when K is a closed *pointed* convex cone, there exists $y \in \mathbb{R}^n$ verifying $\langle y, x\rangle > 0$ for all $x \in K \setminus \{0\}$. Use part (a) to conclude proof of Theorem 2.2.

11. For each of the following sets: show that it is a closed convex pointed cone with nonempty interior, identify the extreme rays and give a simple expression for the dual cone:

    (a) $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \ \forall i = 1, \dots, n\}$

    (b) $\mathbf{Q}^3 = \{(x, t) \in \mathbb{R}^2 \times \mathbb{R}_+ : \|x\|_2 \leq t\}$

    (c) $K = \{(x, y, z) \in \mathbb{R}^2_+ \times \mathbb{R} : \sqrt{xy} \geq |z|\}$

    Show that there is a linear invertible map $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $A(\mathbf{Q}^3) = K$.