## Example class 2

1. Consider the optimization problem:

$$\underset{x\in\mathbb{R}^n, y\in\mathbb{R}^m}{\text{minimise}} \quad \langle c,x\rangle + \langle d,y\rangle \quad \text{s.t.} \quad Fx+Gy=b, \quad x\geq 0 \tag{1}$$

   where $F,G,b,c,d$ are of appropriate sizes, i.e., $F\in\mathbb{R}^{k\times n}$, $G\in\mathbb{R}^{k\times m}$, $b\in\mathbb{R}^k$ and $c\in\mathbb{R}^n, d\in\mathbb{R}^m$. Show how to put (1) into linear programming standard form (Equation (2) in Lecture 3).

2. Let $A\in\mathbf{S}^n_+$ and $u\in\mathbb{R}^n$. Show that $u^TAu=0 \iff u\in\ker(A)$.

3. Let $A\in\mathbf{S}^n$ and $R$ an invertible $n\times n$ matrix. Show that $A\succeq 0 \iff R^TAR\succeq 0$ and $A\succ 0 \iff R^TAR\succ 0$.

4. (Schur complement) Show that

$$\begin{bmatrix} A & B\\ B^T & C\end{bmatrix}\succ 0 \iff A\succ 0 \text{ and } C-B^TA^{-1}B\succ 0 \tag{2}$$

5. (Operator monotonicity of inverse function) Recall that we use the notation $A\succ B$ for $A-B\succ 0$. Show that if $A\succ B\succ 0$ then $A^{-1}\prec B^{-1}$ (*Hint: start with the case* $B=I$ *(identity matrix) then use the fact that* $A\succ B$ *if and only* $B^{-1/2}AB^{-1/2}\succ I$).

6. (Schur product theorem) Let $A,B\in\mathbf{S}^n$ and assume that $A\succeq 0$ and $B\succeq 0$. Show that $A\odot B\succeq 0$ where $A\odot B$ is the *entrywise product* of $A$ and $B$, i.e., $(A\odot B)_{ij}=A_{ij}B_{ij}$ (*Hint: start with the case where* $A$ *has rank one*).

7. Compute the duals of the following problems:

   (a) $\underset{x,y\in\mathbb{R}}{\text{minimise}}\; 2x+y$ s.t. $\begin{bmatrix} 1-x & y\\ y & 1+x\end{bmatrix}\succeq 0$

   (b) $\underset{X\in\mathbf{S}^n}{\text{minimise}}\; \mathrm{Tr}(CX)$ s.t. $X_{ii}=1\;\forall i=1,\dots,n,\;\; X\succeq 0$ ($C$ is a fixed matrix in $\mathbf{S}^n$)

   (c) $\underset{z\in\mathbb{R}^n, y\in\mathbb{R}^m}{\text{maximise}}\; \langle b,y\rangle$ s.t. $c=z+\mathcal{A}^*(y), z\in K^*$ ($\mathcal{A}:\mathbb{R}^n\to\mathbb{R}^m$ linear, $b\in\mathbb{R}^m$, $c\in\mathbb{R}^n$ are fixed).

8. Give an example of a proper cone $K$ and linear map $M$ such that $M(K)$ is not closed.

9. Consider the optimization problem $\min_{x\in\mathbb{R}^n}\|x\|_2$ s.t. $Ax=b, x\geq 0$. Here $A\in\mathbb{R}^{m\times n}, b\in\mathbb{R}^m$ are fixed. Show that this problem can be expressed as a semidefinite program. [*Hint: Express the constraint* $\|x\|_2^2\leq t^2$ *as a semidefinite programming constraint*].

10. Let $M_1,\dots,M_k$ be fixed $n\times n$ symmetric matrices. Consider the optimization problem: $\min_x\; \lambda_{\max}\left(\sum_{i=1}^k x_iM_i\right)$ s.t. $Ax=b$ where $A$ and $b$ are fixed and $\lambda_{\max}$ denotes the largest eigenvalue. Show that it can be expressed as a semidefinite program.

11. Consider the following optimization problem which arises in experiment design (statistics):

$$\text{minimize} \quad \text{Trace}\left[\left(\sum_{i=1}^{k} x_i M_i\right)^{-1}\right] \quad \text{s.t.} \quad x \geq 0, \sum_{i=1}^{k} x_i = 1 \tag{3}$$

where $M_1, \ldots, M_k$ are fixed positive definite matrices. Show that the problem above can be expressed as a semidefinite program. [*Hint: Use the Schur complement lemma to give a semidefinite formulation of the constraint* $\mathrm{Tr}(A^{-1}) \leq t$]

12. (Nesterov's $2/\pi$ result) Let $A$ be a real symmetric matrix of size $n \times n$, and consider the following binary quadratic optimisation problem:

$$\text{maximise} \quad x^T A x \quad : \quad x \in \{-1, 1\}^n. \tag{4}$$

Let $v^*$ be the optimal value of (4).

(a) Consider the semidefinite program:

$$\text{maximise} \quad \mathrm{Tr}(AX) \quad : \quad X \succeq 0 \text{ and } X_{ii} = 1,\ \forall i = 1, \ldots, n. \tag{5}$$

Let $p^*_{SDP}$ be the optimal value of (5). Show that $v^* \leq p^*_{SDP}$.

**From now on we are going to assume that $A$ is positive semidefinite.** The purpose of the rest of this problem is to show that $\frac{2}{\pi} p^*_{SDP} \leq v^*$. To prove this inequality, we will use a "randomised rounding" scheme similar to the one we saw in lecture for the maximum cut problem.

(b) Let $X$ be the optimal solution (5) and let $v_1, \ldots, v_n \in \mathbb{R}^r$ with $r = \mathrm{rank}(X)$ such that $X_{ij} = \langle v_i, v_j \rangle$ for all $i, j = 1, \ldots, n$. Define the random variable $y \in \{-1, 1\}^n$ as follows:

$$y_i = \mathrm{sign}(\langle v_i, Z \rangle)$$

where $Z$ is a standard Gaussian variable on $\mathbb{R}^r$. We saw in lecture that

$$\mathbb{E}[y_i y_j] = \frac{2}{\pi} \arcsin(X_{ij}) \quad \forall 1 \leq i, j \leq n,$$

which you can use without proof. Show that:

$$v^* \geq \mathbb{E}[y^T A y] = \frac{2}{\pi} \mathrm{Tr}(A \arcsin[X]).$$

where $\arcsin[X]$ is the matrix obtained by applying the arcsin function to each entry of $X$, i.e., $\arcsin[X]_{ij} = \arcsin(X_{ij})$.

(c) Recall the Schur product theorem:

> *Schur product theorem*: If $P \succeq 0$ and $Q \succeq 0$ then $P \odot Q \succeq 0$ where $P \odot Q$ is the entrywise product of $P$ and $Q$.

Use the Schur product theorem (without proof) to show that if $X \succeq 0$ then $\arcsin[X] - X \succeq 0$. [*Hint: Use the fact that* $\arcsin(x) = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} x^{2k+1}$ *for* $x \in [-1, 1]$].

(d) Using the positive semidefinite assumption on $A$ show then that $\mathrm{Tr}(A \arcsin[X]) \geq \mathrm{Tr}(AX)$. Conclude that $v^* \geq \frac{2}{\pi} p^*_{SDP}$.