Example class 3

1. The *chromatic number* of a graph G, denoted $\chi(G)$, is the smallest number of colors that are needed to color its vertices in such a way that no two adjacent vertices have the same color. Show that for any graph G we have $\vartheta(G) \leq \chi(\bar{G})$ where $\vartheta(G)$ is the Lovász theta number of G, and $\bar{G}$ is the complement graph of G.

2. Write a semidefinite program that computes the minimum, over $\mathbb{R}$, of the polynomial $p(x) = x^4 + 3x^3 - x^2 + x - 1$. Implement and solve your semidefinite program using CVX.

3. Show that a polynomial $p \in \mathbb{R}[x]$ satisfies $p(x) \geq 0$ for all $x \in [0, \infty)$ if and only if there exist $s_1, s_2 \in \mathbb{R}[x]$ sums-of-squares such that

$$p(x) = s_1(x) + xs_2(x)$$

   with the following degree bounds: $\deg s_1 \leq 2d$ and $\deg s_2 \leq 2d - 2$ if $\deg p = 2d$ (even); and $\deg(s_1) \leq 2d$ and $\deg(s_2) \leq 2d$ if $\deg(p) = 2d + 1$ (odd).

4. Let $a \leq b$. Show that a polynomial $p \in \mathbb{R}[x]$ with even degree $\deg p = 2d$ satisfies $p(x) \geq 0$ on $[a, b]$ if and only if there exist $s_1, s_2 \in \mathbb{R}[x]$ sums-of-squares with $\deg s_1 \leq 2d$ and $\deg s_2 \leq 2d - 2$ such that

$$p(x) = s_1(x) + (b - x)(x - a)s_2(x).$$

   When $\deg p = 2d + 1$ (odd) show that $p(x) \geq 0$ on $[a, b]$ if and only if there exist polynomials $s_1, s_2 \in \mathbb{R}[x]$ sums-of-squares with $\deg s_1 \leq 2d$ and $\deg s_2 \leq 2d$ such that

$$p(x) = (x - a)s_1(x) + (b - x)s_2(x).$$

5. For $p \in \mathbb{R}[x]$ we let $\|p\|_\infty = \max_{x \in [-1,1]} |p(x)|$. Given an integer $n \geq 1$ we are interested in finding the minimum of $\|p\|_\infty$ over all *monic* polynomials p of degree n (recall that a polynomial is called monic if its leading coefficient is equal to 1, where the leading coefficient is the coefficient of the monomial x^n if $n = \deg(p)$). Show how to formulate this problem using the cone of nonnegative polynomials, and solve it using CVX. What optimal values do you get for different choices of n? Can you recognise the polynomial that achieves the optimal value?

6. Show how to formulate the cone of *convex* polynomials using the cone of nonnegative polynomials.

7. Let $y = (y_0, \ldots, y_{2d}) \in \mathbb{R}^{2d+1}$. Show that the solution to the following problem is either $-\infty$ or 0, and that the solution is 0 precisely when $y \in P_{2d}^*$:

$$\underset{p \in \mathbb{R}^{2d+1}, M \in \mathbf{S}^{d+1}}{\text{minimise}} \langle p, y \rangle \quad \text{s.t.} \quad \sum_{\substack{0 \leq i,j \leq d \\ i+j=k}} M_{ij} = p_k, M \succeq 0. \tag{1}$$

   Using strong duality show that $y \in P_{2d}^*$ if and only if $H(y) \succeq 0$.

8. Find the extreme rays of the cone P_{2d} of nonnegative univariate polynomials of degree $2d$.

9. (a) Show that if $p \in \mathbb{R}[x_1, \dots, x_n]$ is nonnegative on $\mathbb{R}^n$ then it has even degree.
   (b) Show that if $p = \sum_k q_k^2$ on $\mathbb{R}^n$ then necessarily $\deg q_k \leq (\deg p)/2$.

10. (a) Show that the cone $P_{n,2d}$ of nonnegative polynomials in n variables of degree $2d$ is a proper cone.
    (b) Show that the cone $\Sigma_{n,2d}$ of sum-of-squares polynomials in n variables of degree $2d$ is a proper cone. [*Hint: you can use Carathéodory theorem without proof: if* $p \in \mathrm{cone}(a_1, \dots, a_M) \subset \mathbb{R}^D$ *then there is a subset* S *of* $\{1, \dots, M\}$ *of size at most* D *such that* $p \in \mathrm{cone}(a_i : i \in S)$].

11. (Based on [Ble15]) Let $s(x) = x_1 + \cdots + x_n$.
    (a) Show that the function $f(x) = (n - s(x))(n - 2 - s(x))$ is nonnegative on $\{-1, 1\}^n$.
    (b) Show that f is not 1-sos on $\{-1, 1\}^n$.
    (c) Show that f is 2-sos on $\{-1, 1\}^n$ [*Hint: what is* $(1 - x_i - x_j + x_i x_j)^2$ *?*]

References

[Ble15] G. Blekherman. Final homework in course "Real Algebraic Geometry and Optimization" at Georgia Tech, 2015. `https://sites.google.com/site/grrigg/home/real-algebraic-geometry-and-optimization`. 2