Example class 3

1. The *chromatic number* of a graph G, denoted $\chi(G)$, is the smallest number of colors that are needed to color its vertices in such a way that no two adjacent vertices have the same color. Show that for any graph G we have $\vartheta(G) \leq \chi(\bar{G})$ where $\vartheta(G)$ is the Lovász theta number of G, and $\bar{G}$ is the complement graph of G.

   Solution: We use the following definition of $\vartheta(G)$ from Lecture 9:

$$\begin{array}{rlll} \vartheta(G) = & \text{min.} & Z_{00} & \\ & \text{s.t.} & Z_{ii} = 1 & \forall i \in V \\ & & Z_{ij} = 0 & \forall ij \in \bar{E} \\ & & \begin{bmatrix} Z_{00} & \mathbf{1}^T \\ \mathbf{1} & Z \end{bmatrix} \succeq 0 & \end{array} \tag{1}$$

   Assume there is a coloring of $\bar{G}$ in k colors. Define $Z_{00} = k$ and the matrix $Z \in \mathbf{S}^n$ by:

$$Z_{ij} = \begin{cases} 1 & \text{if } i \text{ have } j \text{ have the same colors} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

   We claim that the pair (Z_{00}, Z) is feasible for (1). This will show that $\vartheta(G) \leq \chi(\bar{G})$ as desired. Note that $Z_{ij} = 0$ if $ij \in \bar{E}$ by definition of a coloring of $\bar{G}$. It remains to show that the matrix $\left[\begin{smallmatrix} k & \mathbf{1}^T \\ \mathbf{1} & Z \end{smallmatrix}\right] \succeq 0$. Using Schur complements it is sufficient to check that $Z - k^{-1}\mathbf{1}\mathbf{1}^T \succeq 0$

   Note that any coloring of $\bar{G}$ with k colors induces a *partition* on its vertices $V = S_1 \cup \cdots \cup S_k$ where S_c are the vertices of V with color $c \in \{1, \ldots, k\}$. Let $\mathbf{1}_{S_c}$ be the indicator vector for S_c. Then note that the matrix Z defined in Equation (2) can be written as $Z = \sum_{c=1}^k \mathbf{1}_{S_c} \mathbf{1}_{S_c}^T$. Also note that since the sets $S_1, \ldots, S_k$ form a partition of $V = \{1, \ldots, n\}$ we have $\sum_{c=1}^k \mathbf{1}_{S_c} = \mathbf{1}$. Now showing that $Z - k^{-1}\mathbf{1}\mathbf{1}^T \succeq 0$ corresponds to showing that:

$$k \sum_{c=1}^k u_c u_c^T - \left(\sum_{c=1}^k u_c\right)\left(\sum_{c=1}^k u_c\right)^T \succeq 0 \tag{3}$$

   where $u_c = \mathbf{1}_{S_c}$. Inequality (3) can be verified easily by forming the quadratic form of the left-hand side and using Cauchy-Schwarz (the inequality is true for any family of vectors $\{u_1, \ldots, u_k\}$). This completes the proof.

2. Write a semidefinite program that computes the minimum, over $\mathbb{R}$, of the polynomial $p(x) = x^4 + 3x^3 - x^2 + x - 1$. Implement and solve your semidefinite program using CVX.

   Solution: The semidefinite program we write is

$$\max \gamma \quad : \quad Q \succeq 0, \begin{cases} Q_{00} = -1 - \gamma \\ 2Q_{01} = 1 \\ 2Q_{02} + Q_{11} = -1 \\ 2Q_{12} = 3 \\ Q_{22} = 1. \end{cases}$$

   More explicitly this gives

$$\max \gamma \quad : \begin{bmatrix} -1-\gamma & 1/2 & a \\ 1/2 & b & 3/2 \\ a & 3/2 & 1 \end{bmatrix} \succeq 0, \quad 2a + b = -1$$

We can implement this in CVX using the following code:

```
cvx_begin sdp
    variables g a b
    maximize g
    subject to
       [-1-g   1/2   a;
        1/2     b   3/2;
         a     3/2   1] >= 0;
        2*a + b == -1;
cvx_end
```

The value we get is ≈ -17.56.

3. Show that a polynomial $p \in \mathbb{R}[x]$ satisfies $p(x) \geq 0$ for all $x \in [0, \infty)$ if and only if there exist $s_1, s_2 \in \mathbb{R}[x]$ sums-of-squares such that

$$p(x) = s_1(x) + xs_2(x)$$

with the following degree bounds: $\deg s_1 \leq 2d$ and $\deg s_2 \leq 2d - 2$ if $\deg p = 2d$ (even); and $\deg(s_1) \leq 2d$ and $\deg(s_2) \leq 2d$ if $\deg(p) = 2d + 1$ (odd).

Solution: We can proceed by induction on the degree d of the polynomial. If $d = 0$ the result is trivial. Consider a polynomial p of degree d that satisfies $p(x) \geq 0$ for all $x \in [0, \infty)$. Note that the only real roots of p that are allowed to have odd multiplicity must be in $(-\infty, 0]$.

- If p has no real roots in $(-\infty, 0]$ then p is globally nonnegative and is a sum-of-squares.
- If p has a root at $-a$ where $a \geq 0$ then note that $p(x)/(x + a)$ is still nonnegative on $[0, \infty)$. Since $p(x)/(x + a)$ is a polynomial of degree $d - 1$ we can use the induction hypothesis to say that

$$p(x)/(x + a) = s_1(x) + xs_2(x)$$

where s_1, s_2 are sums of squares with $\deg s_1 \leq 2\lfloor (d-1)/2 \rfloor$ and $\deg s_1 \leq 2\lceil (d-1)/2 \rceil - 2$. One can verify then that

$$p(x) = \tilde{s}_1(x) + x\tilde{s}_2(x)$$

where

$$\begin{cases} \tilde{s}_1(x) = as_1(x) + x^2 s_2(x) \\ \tilde{s}_2(x) = s_1(x) + as_2(x) \end{cases}$$

which are both sums of squares. Also note that $\deg \tilde{s}_1 \leq \deg s_2 + 2 \leq 2\lceil (d-1)/2 \rceil = 2\lfloor d/2 \rfloor$ and $\deg \tilde{s}_2 \leq \deg s_1 \leq 2\lfloor (d-1)/2 \rfloor = 2\lceil d/2 \rceil - 2$ as desired.

4. Let $a \leq b$. Show that a polynomial $p \in \mathbb{R}[x]$ with even degree $\deg p = 2d$ satisfies $p(x) \geq 0$ on $[a, b]$ if and only if there exist $s_1, s_2 \in \mathbb{R}[x]$ sums-of-squares with $\deg s_1 \leq 2d$ and $\deg s_2 \leq 2d - 2$ such that

$$p(x) = s_1(x) + (b - x)(x - a)s_2(x).$$

When $\deg p = 2d + 1$ (odd) show that $p(x) \geq 0$ on $[a, b]$ if and only if there exist polynomials $s_1, s_2 \in \mathbb{R}[x]$ sums-of-squares with $\deg s_1 \leq 2d$ and $\deg s_2 \leq 2d$ such that

$$p(x) = (x - a)s_1(x) + (b - x)s_2(x).$$

Solution: We assume $[a, b] = [-1, 1]$ for simplicity. We prove the result by induction on the degree d. Assume p is a polynomial of degree d nonnegative on $[-1, 1]$. Note that any root in $(-1, 1)$ must have even multiplicity.

- If p has no real root outside $(-1,1)$ then $\deg p$ must be even and p is actually a sum-of-squares.
- Assume p has a real root at $-r \le -1$. Then $p(x)/(x+r)$ is still nonnegative on $[-1,1]$. Since $\deg(p(x)/(x+r)) = d-1$ we can use induction.
  - If d is odd then $d-1$ is even and so by the induction hypothesis we know that

$$p(x)/(x+r) = s_1(x) + (1-x^2)s_2(x)$$

    where s_1, s_2 are sums of squares with $\deg s_1 \le d-1$ and $\deg s_2 \le d-3$. One can verify then that

$$p(x) = (1-x)\tilde{s_1}(x) + (1+x)\tilde{s_2}(x)$$

    where

$$\begin{cases} \tilde{s_1}(x) = (1+x)^2 s_2(x) + \frac{r-1}{2}(s_1(x) + (1+x)^2 s_2(x)) \\ \tilde{s_2}(x) = s_1(x) + \frac{r-1}{2}(s_1(x) + (1-x)^2 s_2(x)) \end{cases}$$

    are sums of squares. Note that $\deg \tilde{s_1} \le d-1$ and $\deg \tilde{s_2} \le d-1$ as desired.
  - If d is even then $d-1$ is odd and the induction hypothesis tells us that

$$p(x)/(x+r) = (1-x)s_1(x) + (1+x)s_2(x)$$

    where s_1, s_2 are sums of squares with $\deg s_1, \deg s_2 \le d-2$. One can verify then that

$$p(x) = \tilde{s}_1(x) + (1-x^2)\tilde{s}_2(x)$$

    where

$$\begin{cases} \tilde{s_1}(x) = (1+x)^2 s_2(x) + \frac{r-1}{2}((x-1)^2 s_1(x) + (x+1)^2 s_2(x)) \\ \tilde{s_2}(x) = s_1(x) + \frac{r-1}{2}(s_1(x) + s_2(x)) \end{cases}$$

    are sums of squares. Note that $\deg \tilde{s_1} \le d$ and $\deg \tilde{s_2} \le d-2$.
- If p has a real root at $r \ge 1$ then by doing the change of variables $x \leftrightarrow -x$ we can reduce it to the previous case.

It remains to do the base case $d = 0$ which is trivial.

5. For $p \in \mathbb{R}[x]$ we let $\|p\|_\infty = \max_{x\in[-1,1]} |p(x)|$. Given an integer $n \ge 1$ we are interested in finding the minimum of $\|p\|_\infty$ over all *monic* polynomials p of degree n (recall that a polynomial is called monic if its leading coefficient is equal to 1, where the leading coefficient is the coefficient of the monomial x^n if $n = \deg(p)$). Show how to formulate this problem using the cone of nonnegative polynomials, and solve it using CVX. What optimal values do you get for different choices of n? Can you recognise the polynomial that achieves the optimal value?

Solution: We can formulate our problem as follows:

$$\begin{array}{ll} \underset{t\in\mathbb{R}, p_0,\ldots,p_n\in\mathbb{R}}{\text{minimise}} & t \\ \text{subject to} & t + p \text{ nonnegative on } [-1,1] \\ & t - p \text{ nonnegative on } [-1,1] \\ & p_n = 1 \end{array} \tag{4}$$

where we abbreviated p for the polynomial $p(x) = p_0 + p_1 x + \cdots + p_n x^n$. Using the characterisation of nonnegative polynomials on $[-1,1]$ we can write this problem as, assuming n is even:

$$\begin{array}{ll} \text{minimise} & t \\ \text{subject to} & t + p = s_1 + (1-x^2)s_2 \\ & t - p = \tilde{s_1} + (1-x^2)\tilde{s_2} \\ & p_n = 1 \\ & s_1, s_2, \tilde{s_1}, \tilde{s_2} \text{ sums of squares} \end{array} \tag{5}$$

This problem can be implemented in CVX as follows:

```
n = 4;
cvx_begin
    variable t
    variable p(n+1)
    variable s1(n+1)
    variable s2(n-1)
    variable s1t(n+1)
    variable s2t(n-1)
    minimize t
    subject to
    [zeros(n,1); t] - p == s1 + conv( [-1 ; 0 ; 1] , s2 );
    [zeros(n,1); t] + p == s1t + conv( [-1 ; 0 ; 1] , s2t );
        p(1) == 1; % Monic polynomial constraint -- the convention in Matlab
                   % is that the leading coefficient appears first
        s1 == nonneg_poly_coeffs(n);
        s2 == nonneg_poly_coeffs(n-2);
        s1t == nonneg_poly_coeffs(n);
        s2t == nonneg_poly_coeffs(n-2);
cvx_end
```

The solution of the problem is $1/2^{n-1}$ and the optimal polynomial p is the n'th Chebyshev polynomial. See https://en.wikipedia.org/wiki/Chebyshev_polynomials#Minimal_.E2.88.9E-norm

6. Show how to formulate the cone of *convex* polynomials using the cone of nonnegative polynomials.

Solution: A polynomial is convex if its second derivative is nonnegative everywhere. Using the fact that

$$\frac{d^2}{dx^2}p(x) = \sum_{k=2}^{d} k(k-1)p_k x^{k-2}$$

we have that p'' is convex if and only if $(2p_2, 6p_3, 12p_4, \dots, d(d-1)p_d) \in P_{d-2}$ where $d = \deg(p)$.

7. Let $y = (y_0, \dots, y_{2d}) \in \mathbb{R}^{2d+1}$. Show that the solution to the following problem is either $-\infty$ or 0, and that the solution is 0 precisely when $y \in P_{2d}^*$:

$$\underset{p\in\mathbb{R}^{2d+1}, M\in\mathbf{S}^{d+1}}{\text{minimise}} \langle p, y\rangle \quad \text{s.t.} \quad \sum_{\substack{0\le i,j\le d\\ i+j=k}} M_{ij} = p_k, M \succeq 0. \tag{6}$$

Using strong duality show that $y \in P_{2d}^*$ if and only if $H(y) \succeq 0$.

Solution: The solution is zero when $y \in P_{2d}^*$. If $y \notin P_{2d}^*$ then by definition there is $p_0 \in P_{2d}$ such that $\langle p_0, y\rangle < 0$. But then $p = tp_0$ is feasible for (6) for any $t \ge 0$ and $\langle p, y\rangle \to -\infty$ as $t \to +\infty$.

Let's compute the dual of (6). If we let λ_k be the dual variables for the equality constraints and Z be the dual variable for the positive semidefinite constraint, the inequalities we can infer from the constraints are

$$\sum_{k=0}^{d} \lambda_k (p_k - \sum_{i,j:i+j=k} M_{ij}) + \langle Z, M\rangle \ge 0. \tag{7}$$

with $Z \succeq 0$. Observe that $\sum_{k=0}^{d} \lambda_k \sum_{i,j:i+j=k} M_{ij} = \langle H(\lambda), M\rangle$ where $H(\lambda) = [\lambda_{i+j}]_{0\le i,j\le d}$. Thus we can rewrite (7) as:

$$\langle \lambda, p\rangle + \langle Z - H(\lambda), M\rangle \ge 0.$$

Since we are interested in the cost function $\langle y, p\rangle$ we want $\lambda = y$ and $Z - H(\lambda) = 0$. Thus the dual problem is

$$\text{maximise} \quad 0 \quad \text{s.t.} \quad \lambda = y, \;\; Z - H(\lambda) = 0, \;\; Z \succeq 0.$$

This problem can obviously be simplified to the following trivial problem with no variables:

$$\text{maximise} \quad 0 \quad \text{s.t.} \quad H(y) \succeq 0 \tag{8}$$

The problem (6) can easily be shown to be strictly feasible for example by taking $M = I_n$ and the corresponding p which will be here $p(x) = 1 + x^2 + x^4 + \cdots + x^{2d}$. Thus, by strong duality we know that the optimal value of (6) and (8) are equal. We know that the value of (6) is equal to 0 if and only if $y \in P_{2d}^*$. The value of (8) is 0 if and only if $H(y) \succeq 0$ (otherwise it is infeasible and its value is $-\infty$). Thus this shows that $y \in P_{2d}^*$ if and only if $H(y) \succeq 0$.

8. Find the extreme rays of the cone P_{2d} of nonnegative univariate polynomials of degree $2d$.

Solution: The extreme rays of P_{2d} correspond to nonnegative polynomials with *real roots*; i.e., of the form $p(x) = \prod_{i=1}^k (x - r_i)^{2m_i}$ where $r_1 < r_2 < \cdots < r_k$ and $2m_1 + 2m_2 + \cdots + 2m_k = 2d$.

To see that any such polynomial is extreme, assume $p = p_1 + p_2$ where $p_1, p_2 \geq 0$. Then p_1, p_2 must vanish at $r_1, \ldots, r_k$; furthermore since $p_1, p_2 \geq 0$, each r_i must be a root with even multiplicity of both p_1 and p_2. So we can write $p_1(x) = \prod_{i=1}^k (x - r_i)^2 q_1(x)$ and $p_2(x) = \prod_{i=1}^k (x - r_i)^2 q_2(x)$ for some polynomials $q_1, q_2 \geq 0$. Thus if we define $q(x) = \prod_{i=1}^k (x - r_i)^{2(m_i - 1)}$ we get $q(x) = q_1(x) + q_2(x)$. Using again the same idea we can keep "peeling off" terms from q until we get a constant polynomial $r = r_1 + r_2$ with $r_1, r_2 \geq 0$. By moving up again we see that p_1 and p_2 must be nonnegative multiples of p. (Another way of writing the same argument is via induction on degree).

We now show that any other polynomial $p \in P_{2d}$ is not extreme. Let p be a nonnegative polynomial that has at least one complex nonreal root z. We know that $p(x) = q(x)(x - z)(x - \bar{z})$ for some nonnegative polynomial q. But then if we let $a = \Re[z]$ and $b = \Im[z]$ then $p(x) = q(x)|x - z|^2 = q(x)((x - a)^2 + (x - b)^2) = q_1(x) + q_2(x)$ where $q_1(x) = q(x)(x - a)^2$ and $q_2(x) = q(x)(x - b)^2$ are both nonnegative and not multiples of q.

9. (a) Show that if $p \in \mathbb{R}[x_1, \ldots, x_n]$ is nonnegative on $\mathbb{R}^n$ then it has even degree.

   (b) Show that if $p = \sum_k q_k^2$ on $\mathbb{R}^n$ then necessarily $\deg q_k \leq (\deg p)/2$.

   Solution:

   (a) Let $d = \deg p$ and write $p(\mathbf{x}) = \sum_{\alpha, |\alpha| = d} c_\alpha \mathbf{x}^\alpha + \sum_{\alpha: |\alpha| < d} c_\alpha \mathbf{x}^\alpha = p_0(\mathbf{x}) + p_1(\mathbf{x})$ where $p_0(\mathbf{x})$ consists of the monomials of degree exactly d and $p_1(\mathbf{x})$ the other monomials. Since $p_0 \neq 0$ there exists $\mathbf{a} \in \mathbb{R}^n$ such that $p_0(\mathbf{a}) \neq 0$. Observe that the univariate polynomial (in t) $p(t\mathbf{a})$ is nonnegative and has degree $d = \deg p$. This implies that d must be even.

   (b) Let $\deg p = 2d$ and assume $p = \sum_k q_k^2$. Let $D = \max_k \deg q_k$. By expanding $\sum_k q_k^2$ we see that it will have terms of degree $2D$ with positive coefficients. Thus $2D \leq \deg p = 2d$.

10. (a) Show that the cone $P_{n,2d}$ of nonnegative polynomials in n variables of degree $2d$ is a proper cone.

    (b) Show that the cone $\Sigma_{n,2d}$ of sum-of-squares polynomials in n variables of degree $2d$ is a proper cone. [*Hint: you can use Carathéodory theorem without proof: if* $p \in \operatorname{cone}(a_1, \ldots, a_M) \subset \mathbb{R}^D$ *then there is a subset* S *of* $\{1, \ldots, M\}$ *of size at most* D *such that* $p \in \operatorname{cone}(a_i : i \in S)$].

    Solution:

(a) By definition

$$P_{n,2d} = \{p \in \mathbb{R}[\mathbf{x}]_{\leq 2d} : p(x) \geq 0 \; \forall x \in \mathbb{R}^n\}.$$

For each $x \in \mathbb{R}^n$, the set $H_x = \{p \in \mathbb{R}[\mathbf{x}]_{\leq 2d} : p(x) \geq 0\}$ is a closed halfspace. Thus $P_{n,2d}$ is closed and convex as an intersection of closed convex sets. It is pointed because if $p \in P_{n,2d}$ and $p \in -P_{n,2d}$ then $p(x) = 0$ for all $x \in \mathbb{R}^n$ which implies that p is the zero polynomial. Finally one can check that it has nonempty interior by verifying that $p(\mathbf{x}) = x_1^{2d} + \cdots + x_n^{2d} + 1$ is in the interior. To be precise consider the following norm defined on the space of polynomials of degree at most $2d$:

$$\|q\| = \sum_{|\alpha| \leq 2d} |q_\alpha|.$$

In other words, this is the ℓ_1 norm of the coefficients of q. We will now show that if $\|q\| \leq 1$ then $p + q \in P_{n,2d}$ where $p(\mathbf{x}) = x_1^{2d} + \cdots + x_n^{2d} + 1$. Let thus q such that $\|q\| \leq 1$. Let $\mathbf{x} \in \mathbb{R}^n$ and assume that $|x_i| \leq 1$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned}
p(\mathbf{x}) + q(\mathbf{x}) &= x_1^{2d} + \cdots + x_n^{2d} + 1 + \sum_{|\alpha| \leq 2d} q_\alpha x^\alpha \\
&\geq x_1^{2d} + \cdots + x_n^{2d} + 1 - \sum_{|\alpha| \leq 2d} |q_\alpha||x^\alpha| \\
&\geq x_1^{2d} + \cdots + x_n^{2d} + 1 - \sum_{|\alpha| \leq 2d} |q_\alpha| \\
&\geq x_1^{2d} + \cdots + x_n^{2d} \geq 0
\end{aligned}$$

where in second inequality we used the fact that $\max_i |x_i| \leq 1$ and in the third inequality we used that $\|q\| \leq 1$. Assume now that there is at least one i such that $|x_i| \geq 1$. Let $j = \operatorname{argmax}_i |x_i|$ and note that for any α with $|\alpha| \leq 2d$ we have $|x^\alpha x_j^{-2d}| \leq 1$. This allows us to write:

$$\begin{aligned}
p(\mathbf{x}) + q(\mathbf{x}) &= x_1^{2d} + \cdots + x_n^{2d} + 1 + \sum_{|\alpha| \leq 2d} q_\alpha x^\alpha \\
&= x_j^{2d} \left(1 + \sum_{i \neq j} x_i^{2d} x_j^{-2d} + x_j^{-2d} + \sum_{|\alpha| \leq 2d} q_\alpha x^\alpha x_j^{-2d} \right) \\
&\geq x_j^{2d} \left(1 + \sum_{i \neq j} x_i^{2d} x_j^{-2d} + x_j^{-2d} - \sum_{|\alpha| \leq 2d} |q_\alpha||x^\alpha||x_j^{-2d}| \right) \\
&\geq x_j^{2d} \left(1 + \sum_{i \neq j} x_i^{2d} x_j^{-2d} + x_j^{-2d} - \sum_{|\alpha| \leq 2d} |q_\alpha| \right) \\
&\geq 0
\end{aligned}$$

This proves that $p + q$ is nonnegative everywhere which is what we wanted.[1]

(b) Showing that $\Sigma_{n,2d}$ is convex and pointed is easy. We need to show that it is closed and has nonempty interior.

- Closedness: For convenience we define the following norm on the space of polynomials:

$$\|q\| = \max_{\|\mathbf{x}\|_2 \leq 1} |q(\mathbf{x})|.$$

[1] The reason this proof is a bit lengthy is that we are working with nonhomogeneous polynomials. When working with *homogeneous* polynomials, showing that the cone has nonempty interior is much easier, by simply considering the polynomial $(x_1^2 + \cdots + x_n^2)^d$ and the norm $\|q\| = \max_{\|x\|_2 = 1} |q(x)|$.

It is easy to check that it defines a norm on polynomials. Let (p_k) be a sequence of polynomials in $\Sigma_{n,2d}$ that converges to p. We need to show that $p \in \Sigma_{n,2d}$. For each k we can write:

$$p_k = \sum_{i=1}^{N} q_{ki}^2 \tag{9}$$

where $q_{ki} \in \mathbb{R}[\mathbf{x}]_{\le d}$ and $N = \dim \mathbb{R}[\mathbf{x}]_{\le 2d}$ (by Carathéodory theorem). Let $i \in \{1, \ldots, N\}$ and consider the sequence of polynomials $(q_{ki})_{k\in\mathbb{N}}$. Note that from (9) we get that $q_{ki}^2(\mathbf{x}) \le p_k(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Since $(p_k)_{k\in\mathbb{N}}$ is a bounded sequence (since it is convergent) we can find some $M > 0$ such that $\|p_k\| \le M$ for all k and so we get that for all k, $\|q_{ki}\| \le \sqrt{M}$, i.e., $(q_{ki})_{k\in\mathbb{N}}$ is bounded. We can thus extract from it a convergent subsequence that converges to some $q_i \in \mathbb{R}[\mathbf{x}]_{\le d}$. Taking limits in (9) we get $p = \sum_{i=1}^{N} q_i^2$ as desired.

– Nonempty interior: We know that $\Sigma_{n,2d}$ has the following semidefinite characterization:

$$\Sigma_{n,2d} = \pi(\mathbf{S}_+^{s(n,d)})$$

where

$$\pi(Q)_\gamma = \sum_{\alpha,\beta:\alpha+\beta=\gamma} Q_{\alpha,\beta} \quad \forall \gamma : |\gamma| \le 2d.$$

The map $\pi : \mathbf{S}^{s(n,d)} \to \mathbb{R}[\mathbf{x}]_{\le 2d}$ can easily be seen to be surjective. Thus this implies that if B is any ball $\mathbf{S}^{s(n,d)}$ with nonempty interior then its image under π will have nonempty interior. From this observation we can see that $\pi(I)$ (where I is the identity matrix in $\mathbf{S}^{s(n,d)}$) lives in the interior of $\Sigma_{n,2d}$, because I is in the interior of $\mathbf{S}_+^{s(n,d)}$.

11. (Based on [Ble15]) Let $s(x) = x_1 + \cdots + x_n$.

(a) Show that the function $f(x) = (n - s(x))(n - 2 - s(x))$ is nonnegative on $\{-1, 1\}^n$.

(b) Show that f is not 1-sos on $\{-1, 1\}^n$.

(c) Show that f is 2-sos on $\{-1, 1\}^n$ [*Hint: what is* $(1 - x_i - x_j + x_i x_j)^2$*?*]

Solution:

(a) The function $s(x)$ can only take values $n, n-2, n-4, \ldots, -n$ on $\{-1, 1\}^n$. Thus it follows that $f(x) \ge 0$ for all $x \in \{-1, 1\}^n$.

(b) Assume $f(x) = \sum_k q_k(x)^2$ where $q_k(x)$ are of degree 1, i.e., $q_k(x) = a_{k1}x_1 + \cdots + a_{kn}x_n + b_k$. Since $f(x) = 0$ for $x = (1, \ldots, 1)$ we must have $a_{k1} + \cdots + a_{kn} + b_k = 0$ for all k. Similarly note that $f(x) = 0$ whenever x has exactly one component equal to -1 and so this tells us that for any k, $\epsilon_1 a_{k1} + \cdots + \epsilon_n a_{kn} + b_k = 0$ where $\epsilon \in \{-1, 1\}^n$ has exactly one component equal to -1. For any fixed k this gives us $n + 1$ linear equations in the $a_{k1}, \ldots, a_{kn}, b_k$ and it is easy to see that they imply $a_{k1} = \cdots = a_{kn} = b_k = 0$ for all k. But f is not the all-zero function and so we get a contradiction.

(c) If we expand $f(x)$ in the square-free monomial basis we get

$$f(x) = n(n-1) - 2(n-1)s(x) + 2 \sum_{1 \le i < j \le n} x_i x_j.$$

For any $i < j$ we have $(1 - x_i - x_j + x_i x_j)^2 = 4(1 - x_i - x_j + x_i x_j)$. Thus it is easy to verify that we have

$$f(x) = \frac{1}{2} \sum_{1 \le i < j \le n} (1 - x_i - x_j + x_i x_j)^2.$$

References

[Ble15] G. Blekherman. Final homework in course "Real Algebraic Geometry and Optimization" at Georgia Tech, 2015. https://sites.google.com/site/grrigg/home/real-algebraic-geometry-and-optimization. 7