Exercises for revision class

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1 Diagonally dominant matrices

A matrix $A \in \mathbf{S}^n$ is called *diagonally dominant* if $A_{ii} \geq \sum_{j\neq i} |A_{ij}|$ for all $i = 1, \ldots, n$. Let $\mathcal{D}_n \subset \mathbf{S}^n$ be the set of diagonally dominant matrices.

(a) Show that if A is diagonally dominant then it is positive semidefinite.

(b) Recall the definition of *proper cone*. Show that the set $\mathcal{D}_n$ is a proper cone in $\mathbf{S}^n$.

(c) Show that the extreme rays of $\mathcal{D}_n$ are spanned by the matrices

$$e_i e_i^T \quad (i = 1, \ldots, n) \qquad \text{and} \qquad (e_i \pm e_j)(e_i \pm e_j)^T \quad (1 \leq i < j \leq n).$$

where $e_i \in \mathbb{R}^n$ is the vector with 1 in the i'th component and 0 elsewhere.

2 Euclidean distance matrices

Let $(d_{ij})_{1\leq i<j\leq n}$ be $\binom{n}{2}$ positive numbers. Show that the following two assertions are equivalent:

(i) There exist points $x_1, \ldots, x_n \in \mathbb{R}^k$ (for some k) such that $d_{ij} = \|x_i - x_j\|_2$ for all $i < j$.

(ii) The $n \times n$ symmetric matrix $D = \left[d_{ij}^2\right]_{1\leq i,j\leq n}$ (where $d_{ii} = 0$) is negative semidefinite on the subspace orthogonal to $e = (1, \ldots, 1) \in \mathbb{R}^n$. [*We say that a matrix $A \in \mathbf{S}^n$ is negative semidefinite on a subspace L if $x^T Ax \leq 0$ for all $x \in L$*]

3 Faces of the positive semidefinite cone

(a) Recall the definition of a *face* of a convex set. Let V be a subspace of $\mathbb{R}^n$. Show that

$$F_V = \left\{Y \in \mathbf{S}^n_+ : \operatorname{im} Y \subseteq V\right\}$$

is a face of the positive semidefinite cone. What is its dimension?

(b) Find a $C \in \mathbf{S}^n$ such that $\operatorname{argmin}_{X\in\mathbf{S}^n_+} \langle C, X\rangle = F_V$ (this shows that F_V is a so-called *exposed face* of $\mathbf{S}^n_+$).

(c) Let $X \in \mathbf{S}^n_+$. Show that the smallest face of $\mathbf{S}^n_+$ containing X is $F_{\operatorname{im} X}$.

4 Existence of extreme points

Given a set $C \subseteq \mathbb{R}^n$ we say that C contains a *straight line* if there exists $x \in C$ and $v \in \mathbb{R}^n$ such that $x + tv \in C$ for all $t \in \mathbb{R}$.

(a) Let C be a nonempty closed convex set that does not contain any straight lines. Show that C has an extreme point [*Hint: you can use an argument by induction on the dimension of C, similar to the proof of Theorem 1.2 we did in lecture*].

(b) Conversely, show that if C is a closed convex set with an extreme point then it does not contain any straight lines.

5 Extreme points in linear programming

(a) Recall the definition of *extreme point* of a convex set.

(b) Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and consider the convex set $P = \{x \in \mathbb{R}^n_+ : Ax = b\}$. Show that any extreme point x of P satisfies $|\operatorname{supp}(x)| \leq m$ where $\operatorname{supp}(x) := \{i \in [n] : x_i \neq 0\}$ [*Hint: Show that if x is an extreme point of P then* $\ker(A) \cap \{y \in \mathbb{R}^n : \operatorname{supp}(y) \subseteq \operatorname{supp}(x)\} = \{0\}$].

Use Exercise 4 to show that if P is not empty then it has at least one extreme point.

(c) Use the result of part (b) to prove Carathéodory's theorem:

> *Carathéodory's theorem*: Let $S \subset \mathbb{R}^N$ be a finite set. Then any element of $\operatorname{conv}(S)$ can be expressed as a convex combination of at most $N+1$ points of S.

6 Extreme points in semidefinite programming

Part (a) of this exercise is the analogue of Exercise 5(a) for the case of semidefinite programming.

(a) Let $\mathcal{A} : \mathbf{S}^n \to \mathbb{R}^m$ be a linear map, $b \in \mathbb{R}^m$ and let $C = \{X \in \mathbf{S}^n_+ : \mathcal{A}(X) = b\}$. Show that any extreme point X of C satisfies $r(r+1)/2 \leq m$ where $r = \operatorname{rank} X$ [*Hint: Show that if X is an extreme point of C then* $\ker(\mathcal{A}) \cap \{Y \in \mathbf{S}^n : \operatorname{im}(Y) \subseteq \operatorname{im}(X)\} = \{0\}$].

Use Exercise 4 to show that if C is nonempty then it has at least one extreme point.

(b) Let $A, B \in \mathbf{S}^n$. Use part (a) to show that the set

$$R(A,B) = \{(x^T A x, x^T B x) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$$

is convex. (This set is known as the *numerical range* or *field of values* of the pair (A, B).) [*Hint: consider* $\{(\langle A, X\rangle, \langle B, X\rangle) : X \in \mathbf{S}^n_+\}$].

(c) Prove the following result, known as the *S-lemma*: Let $A, B \in \mathbf{S}^n$ and assume that for any $x \in \mathbb{R}^n$, $x^T A x \geq 0 \Rightarrow x^T B x \geq 0$. Assume furthermore that there exists $z \in \mathbb{R}^n$ such that $z^T A z > 0$. Show that there exists $\lambda \geq 0$ such that $B \succeq \lambda A$.

Give an example of $A, B \in \mathbf{S}^2$ to show that the condition of existence of $z \in \mathbb{R}^n$ such that $z^T A z > 0$ cannot be removed in general.

7 Matrix square root

(a) Let $A, B \succ 0$. Show that if $A^2 \succeq B^2$ then $A \succeq B$ [*Hint: let v be an eigenvector of $A - B$ and consider* $v^T(A+B)(A-B)v$].

(b) Give an example of $A, B \in \mathbf{S}^2_{++}$ such that $A \succeq B$ but $A^2 \not\succeq B^2$.

8 Newton polytope

For a polynomial $p(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \mathbf{x}^\alpha$ we define the Newton polytope of p to be

$$\operatorname{Newton}(p) = \operatorname{conv}\{\alpha \in \mathbb{N}^n : p_\alpha \neq 0\}.$$

(For example the Newton polytope of $p(\mathbf{x}) = x_1^3x_2 + 2x_1x_2 - 4x_1x_2^2$ is $\operatorname{conv}\{(3,1),(1,1),(1,2)\} \subset \mathbb{R}^2$.) Show that if

$$p = \sum_i q_i^2$$

then for all i, $\operatorname{Newton}(q_i) \subseteq \frac{1}{2}\operatorname{Newton}(p)$. [*Hint: consider an extreme point of* $\operatorname{conv}(\bigcup_i \operatorname{Newton}(q_i))$].

9 Homogeneous and nonhomogeneous polynomials

A polynomial $p \in \mathbb{R}[x_1,\ldots,x_n]$ is called *homogeneous of degree* d if it only involves monomials of degree exactly d. Given a nonhomogeneous polynomial p of degree d we can *homogenise* it by introducing an additional variable x_0 via

$$\bar{p}(x_0,x_1,\ldots,x_n) = x_0^d p(x_1/x_0,\ldots,x_n/x_0) \tag{1}$$

(a) Show that (1) is well-defined. What is the homogenisation of $p(x_1,x_2) = x_1^2x_2^2 - 2x_1x_2 + 1$?

(b) Show that p is nonnegative if and only if $\bar{p}$ is nonnegative.

(c) Show that p is a sum of squares if and only if $\bar{p}$ is a sum of squares.

(d) Show that if p is a homogeneous polynomial of degree $2d$ and $p = \sum_k q_k^2$ then the q_k must be homogeneous of degree d.

10 A nonnegative polynomial that is not a sum of squares

In lecture we saw the Motzkin polynomial $M(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ which is an explicit example of a nonnegative polynomial that is not a sum of squares in the case $(n,2d) = (2,6)$ (where n is the number of variables and $2d$ the degree). In this exercise we look at a polynomial in 3 variables of degree 4 (i.e., $(n,2d) = (3,4)$) that is nonnegative but not a sum-of-squares. Consider the following polynomial (due to Choi and Lam [CL77]).

$$Q(x,y,z) = x^2y^2 + x^2z^2 + y^2z^2 + 1 - 4xyz.$$

(a) Show that $Q(x,y,z) \geq 0$ for all $(x,y,z) \in \mathbb{R}^2$.

(b) Show that Q is not a sum of squares.

11 Positive and decomposable maps

(Based on exercise 3.178 in [BPT12]) A map $\Lambda : \mathbf{S}^{n_1} \to \mathbf{S}^{n_2}$ is called *positive* if $\Lambda(A) \succeq 0$ whenever $A \succeq 0$.

(a) Show that if Λ has the form $\Lambda(A) = \sum_{i=1}^r P_i^T A P_i$ where $P_1,\ldots,P_r \in \mathbb{R}^{n_1 \times n_2}$ then Λ is positive. Such maps are called *decomposable*.

(b) To any linear map $\Lambda : \mathbf{S}^{n_1} \to \mathbf{S}^{n_2}$ we can consider the polynomial $p(x,y) = y^T\Lambda(xx^T)y$ where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$. Show that Λ is a positive map if and only if p is nonnegative. Show that Λ is decomposable if and only if p is a sum-of-squares.

(c) Consider the following map $\Lambda : \mathbf{S}^3 \to \mathbf{S}^3$ due to M.-D. Choi [Cho75]:

$$\Lambda(A) = 2 \begin{bmatrix} a_{11} + a_{22} & 0 & 0 \\ 0 & a_{22} + a_{33} & 0 \\ 0 & 0 & a_{33} + a_{11} \end{bmatrix} - A.$$

(i) Show that Λ is positive [*Hint: in the case* $a_{33} \geq a_{11}$ *use* $\Lambda(A) = DAD + \left[\begin{smallmatrix} 2a_{22} & -2a_{12} & 0 \\ -2a_{12} & 2a_{33} & 0 \\ 0 & 0 & 2a_{11} \end{smallmatrix}\right]$ *with* $D = \operatorname{diag}(1, 1, -1)$*; then generalise using cyclic symmetry of* Λ].

(ii) Show that Λ is not decomposable. [*Hint: show that the associated polynomial* $p(x, y)$ *is not a sum-of-squares*].

References

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