Exercises for revision class

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1 Diagonally dominant matrices

A matrix $A \in \mathbf{S}^n$ is called *diagonally dominant* if $A_{ii} \geq \sum_{j \neq i} |A_{ij}|$ for all $i = 1, \ldots, n$. Let $\mathcal{D}_n \subset \mathbf{S}^n$ be the set of diagonally dominant matrices.

(a) Show that if A is diagonally dominant then it is positive semidefinite.

(b) Recall the definition of *proper cone*. Show that the set $\mathcal{D}_n$ is a proper cone in $\mathbf{S}^n$.

(c) Show that the extreme rays of $\mathcal{D}_n$ are spanned by the matrices

$$e_i e_i^T \quad (i = 1, \ldots, n) \quad \text{and} \quad (e_i \pm e_j)(e_i \pm e_j)^T \quad (1 \leq i < j \leq n).$$

where $e_i \in \mathbb{R}^n$ is the vector with 1 in the i'th component and 0 elsewhere.

Solution.

(a) For any $x \in \mathbb{R}^n$ we have

$$\begin{aligned} x^T A x = \sum_{i=1}^n A_{ii} x_i^2 + 2 \sum_{i<j} A_{ij} x_i x_j &\overset{(a)}{\geq} \sum_{i=1}^n A_{ii} x_i^2 - \sum_{i<j} |A_{ij}|(x_i^2 + x_j^2) \\ &= \sum_{i=1}^n (A_{ii} - \sum_{j \neq i} |A_{ij}|) x_i^2 \geq 0 \end{aligned}$$

where in (a) we used the arithmetic-geometric mean inequality.

(b) A proper cone is a closed, convex, pointed, full-dimensional cone. That $\mathcal{D}_n$ is closed is easy to see. To prove that $\mathcal{D}_n$ is convex one can use the definition of convexity and use the fact that the absolute value function is convex. Alternatively it is easy to verify that

$$\mathcal{D}_n = \{A \in \mathbf{S}^n : A_{ii} \geq \sum_{j \neq i} \epsilon_{ij} A_{ij} \ \forall \epsilon_{ij} \in \{-1, 1\}, \forall i = 1, \ldots, n\}.$$

Thus $\mathcal{D}_n$ is an intersection of closed halfspaces and is thus a closed convex set. Showing that $\mathcal{D}_n$ is pointed is not difficult. We can show that $\mathcal{D}_n$ has nonempty interior by showing that $I \in \mathcal{D}_n$ where I is the identity matrix. Indeed if we let N be the entrywise infinity norm on $\mathbf{S}^n$ then it is not hard to see that the ball $\{A : N(A - I) \leq 1/n\}$ is in $\mathcal{D}_n$.

(c) We first show that the given matrices are extreme rays: since the given matrices are all rank-one they are extreme rays of $\mathbf{S}^n_+$ and since $\mathcal{D}_n \subseteq \mathbf{S}^n_+$ it follows that they are also extreme rays of $\mathcal{D}_n$.

We now show that they are the only extreme rays. To do this we show that any $A \in \mathcal{D}_n$ can be expressed as a conic combination of the given matrices. Indeed note that we have

$$A = \sum_{i<j} |A_{ij}|(e_i \pm e_j)(e_i \pm e_j)^T + \sum_{i=1}^n (A_{ii} - \sum_{j \neq i} |A_{ij}|) e_i e_i^T$$

where in the first term the sign $\pm$ is "+" if $A_{ij} > 0$ and "−" if $A_{ij} < 0$.

2 Euclidean distance matrices

Let $(d_{ij})_{1\le i<j\le n}$ be $\binom{n}{2}$ positive numbers. Show that the following two assertions are equivalent:

(i) There exist points $x_1,\dots,x_n \in \mathbb{R}^k$ (for some k) such that $d_{ij} = \|x_i - x_j\|_2$ for all $i<j$.

(ii) The $n\times n$ symmetric matrix $D = \left[d_{ij}^2\right]_{1\le i,j\le n}$ (where $d_{ii}=0$) is negative semidefinite on the subspace orthogonal to $e=(1,\dots,1)\in\mathbb{R}^n$. [*We say that a matrix $A\in\mathbf{S}^n$ is negative semidefinite on a subspace L if $x^TAx\le 0$ for all $x\in L$*]

Solution. We first show $(i)\Rightarrow(ii)$. Assume $d_{ij}=\|x_i-x_j\|_2$ for some $x_1,\dots,x_n\in\mathbb{R}^k$. Then $D_{ij}=d_{ij}^2=\|x_i\|^2+\|x_j\|^2-2\langle x_i,x_j\rangle$. Take z orthogonal to $e=(1,\dots,1)$., i.e., $\sum_{i=1}^n z_i=0$. Then

$$z^TDz=\sum_{ij} z_iz_j(\|x_i\|^2+\|x_j\|^2-2\langle x_i,x_j\rangle)\overset{(*)}{=}-2\sum_{ij} z_iz_j\langle x_i,x_j\rangle=-2\left\|\sum_{i=1}^n z_ix_i\right\|^2\le 0.$$

where in $(*)$ we used the fact that $\sum_{i=1}^n z_i=0$.

We now show $(ii)\Rightarrow(i)$. Assume D is negative semidefinite on the subspace orthogonal to $e=(1,\dots,1)\in\mathbb{R}^n$. Let P be the $n\times(n-1)$ matrix whose columns are e_j-e_1 where $j=2,\dots,n$, where e_k is the vector with a one in component k and zeros elsewhere. Note that $\operatorname{im}(P)$ is the subspace orthogonal to e. Thus P^TDP is negative semidefinite and we can thus find vectors $x_2,\dots,x_n$ such that $(P^TDP)_{ij}=-2\langle x_i,x_j\rangle$ for all $i,j\ge 2$ (the reason why we put a constant 2 will be clear later). Define $x_1=0$. We now claim that the vectors $x_1,\dots,x_n$ satisfy $d_{ij}^2=\|x_i-x_j\|^2$ as desired. To show this observe that by definition of P we have $(P^TDP)_{ij}=(e_i-e_1)^TD(e_j-e_1)=d_{ij}^2-d_{i1}^2-d_{j1}^2$. Thus by definition of the vectors $x_2,\dots,x_n$ we have:

$$d_{ij}^2-d_{i1}^2-d_{j1}^2=-2\langle x_i,x_j\rangle\quad\forall i,j\ge 2. \tag{1}$$

Taking $i=j\ge 2$ in (1) tells us that $d_{1i}^2=\|x_i\|^2=\|x_i-x_1\|^2$ for all $i\ge 2$. Then taking $i\ne j$ with $i,j\ge 2$ tells us that $d_{ij}^2=\|x_i\|^2+\|x_j\|^2-2\langle x_i,x_j\rangle=\|x_i-x_j\|^2$ for all $i,j\ge 2$. This completes the proof.

3 Faces of the positive semidefinite cone

(a) Recall the definition of a *face* of a convex set. Let V be a subspace of $\mathbb{R}^n$. Show that

$$F_V=\left\{Y\in\mathbf{S}_+^n : \operatorname{im} Y\subseteq V\right\}$$

is a face of the positive semidefinite cone. What is its dimension?

(b) Find a $C\in\mathbf{S}^n$ such that $\operatorname{argmin}_{X\in\mathbf{S}_+^n}\langle C,X\rangle=F_V$ (this shows that F_V is a so-called *exposed face* of $\mathbf{S}_+^n$).

(c) Let $X\in\mathbf{S}_+^n$. Show that the smallest face of $\mathbf{S}_+^n$ containing X is $F_{\operatorname{im} X}$.

Solution.

(a) A subset F of a convex set C is a *face* if it is convex and if whenever $(a+b)/2\in F$ with $a,b\in C$, then $a,b\in F$. We now proceed to show that F_V is a face of $\mathbf{S}_+^n$.

– We first need to show that F_V is convex. Assume $Y_1, Y_2 \in F_V$. Then it is clear that any convex combination (in fact any linear combination) of Y_1 and Y_2 is in F_V.

– Now assume that $Y \in F_V$ and $Y = (A+B)/2$ where $A, B \in \mathbf{S}^n_+$. We have to show that $A, B \in F_V$. Let x orthogonal to V. Then we have $0 = x^T Y x = (x^T A x + x^T B x)/2$ thus it follows that $x^T A x = x^T B x = 0$ since $A, B \succeq 0$. Again since $A, B \succeq 0$ this also implies that $x \in \ker(A)$ and $x \in \ker(B)$. We have thus shown that $V^\perp \subseteq \ker(A)$ hence $\operatorname{im}(A) \subseteq V$, and similarly for B as desired.

The dimension of F_V is $r(r+1)/2$ where $r = \dim V$.

(b) Take $C \in \mathbf{S}^n$ defined by $C_{|V} = 0$ and $C|_{V^\perp} = I_{V^\perp}$ (where $I_{V^\perp}$ is the identity on $V^\perp$). Since $C \succeq 0$ we have $\langle C, X \rangle \geq 0$ for all $X \succeq 0$. Now we claim that $\langle C, X \rangle = 0$ if and only if $X \in F_V$. Indeed if $X \in F_V$ we have $CX = 0$ since $\operatorname{im}(X) \subseteq V = \ker(C)$ and so $\langle C, X \rangle = \operatorname{Tr}(CX) = 0$. On the other hand if $\operatorname{Tr}(CX) = 0$ then if we let (v_i) be an orthonormal basis of $V^\perp$ so that $C = \sum_i v_i v_i^T$ we get $0 = \operatorname{Tr}(\sum_i v_i v_i^T X) = \sum_i v_i^T X v_i$. Since $X \succeq 0$ we get that necessarily $v_i \in \ker(X)$ for all i. This means that $V^\perp \subseteq \ker(X)$ i.e., $\operatorname{im}(X) \subseteq V$ as desired.

(c) Let $X \in \mathbf{S}^n_+$ and let $r = \operatorname{rank}(X)$. Observe that we can write $X = Q \left[\begin{smallmatrix} X_0 & 0 \\ 0 & 0 \end{smallmatrix}\right] Q^T$ where $X_0 \in \mathbf{S}^r_{++}$ is invertible and Q orthogonal. Using this notation $F_{\operatorname{im} X} = \{Q \left[\begin{smallmatrix} Z & 0 \\ 0 & 0 \end{smallmatrix}\right] Q^T : Z \in \mathbf{S}^r_+\}$. Let F be the smallest closed face containing X. We will show that $F_{\operatorname{im} X} \subseteq F$. Let $Z \in \mathbf{S}^r_+$. Since X_0 is invertible we can find small enough $\epsilon > 0$ such that $X_0 \succeq \epsilon Z$. Now observe that $X = Q \left[\begin{smallmatrix} \epsilon Z & 0 \\ 0 & 0 \end{smallmatrix}\right] Q^T + Q \left[\begin{smallmatrix} X_0 - \epsilon Z & 0 \\ 0 & 0 \end{smallmatrix}\right] Q^T$. Each term in this sum is in the cone $\mathbf{S}^n_+$, thus by definition of face they must be in F. Thus this proves that $Q \left[\begin{smallmatrix} Z & 0 \\ 0 & 0 \end{smallmatrix}\right] Q^T$ is in F. Since this is true for any $Z \in \mathbf{S}^r_+$ we have thus shown that $F_{\operatorname{im} X} \subseteq F$ as desired.

4 Existence of extreme points

Given a set $C \subseteq \mathbb{R}^n$ we say that C contains a *straight line* if there exists $x \in C$ and $v \in \mathbb{R}^n$ such that $x + tv \in C$ for all $t \in \mathbb{R}$.

(a) Let C be a nonempty closed convex set that does not contain any straight lines. Show that C has an extreme point [*Hint: you can use an argument by induction on the dimension of C, similar to the proof of Theorem 1.2 we did in lecture*].

(b) Conversely, show that if C is a closed convex set with an extreme point then it does not contain any straight lines.

Solution.

(a) We use induction on the dimension. It is clear in dimension 1. Assume $C \subset \mathbb{R}^n$ has dimension n. Let $x \in P$ and consider a straight line L that goes through x. The intersection of L and C is a closed interval possibly unbounded with at least one extreme point z that lies on the boundary of C. Let F be a face of C dimension $\leq n-1$ such that $z \in F$ and use the induction hypothesis on F.

(b) We will show the contrapositive. Assume C contains a straight line, i.e., there exists $x \in C$ and v such that $x + tv \in C$ for all $t \in \mathbb{R}$. We will show that for any $z \in C$ we must have $z + tv \in C$ for all $t \in \mathbb{R}$. Indeed for any $t \in \mathbb{R}$ and $s \geq 1$ we have

$$\frac{1}{s}(x + stv) + (1 - \frac{1}{s})z = z + tv + (x - z)/s \in C$$

Letting $s \to \infty$ and using the closedness of C we see that $z + tv \in C$ for any $t \in \mathbb{R}$. It follows from this that C does not have any extreme point.

5 Extreme points in linear programming

(a) Recall the definition of *extreme point* of a convex set.

(b) Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and consider the convex set $P = \{x \in \mathbb{R}^n_+ : Ax = b\}$. Show that any extreme point x of P satisfies $|\operatorname{supp}(x)| \leq m$ where $\operatorname{supp}(x) := \{i \in [n] : x_i \neq 0\}$ [*Hint: Show that if x is an extreme point of P then* $\ker(A) \cap \{y \in \mathbb{R}^n : \operatorname{supp}(y) \subseteq \operatorname{supp}(x)\} = \{0\}$].

Use Exercise 4 to show that if P is not empty then it has at least one extreme point.

(c) Use the result of part (b) to prove Carathéodory's theorem:

> *Carathéodory's theorem*: Let $S \subset \mathbb{R}^N$ be a finite set. Then any element of $\operatorname{conv}(S)$ can be expressed as a convex combination of at most $N + 1$ points of S.

Solution.

(a) A point x is an extreme point of a convex set C if whenever $x = \lambda a + (1-\lambda)b$ with $0 < \lambda < 1$ and $a, b \in C$ it holds $a = b = x$.

(b) Let x be an extreme point of P. As indicated in the hint we will prove that $\ker(A) \cap \{y \in \mathbb{R}^n : \operatorname{supp}(y) \subseteq \operatorname{supp}(x)\} = \{0\}$. Note that this will prove the desired result: indeed $\{y \in \mathbb{R}^n : \operatorname{supp}(y) \subseteq \operatorname{supp}(x)\}$ is a subspace of dimension $|\operatorname{supp}(x)|$ and so for the intersection with $\ker(A)$ to be $\{0\}$ we must have $|\operatorname{supp}(x)| \leq n - \dim(\ker(A)) = \dim \operatorname{im}(A) \leq m$.

We now prove the claim. Assume y satisfies $Ay = 0$ and $\operatorname{supp}(y) \subseteq \operatorname{supp}(x)$. Since $x_i > 0$ for $i \in \operatorname{supp}(x)$ we get that $x \pm \epsilon y \in P$ for small enough $\epsilon > 0$. Since x is an extreme point it must be that $y = 0$.

To show that P has at least one extreme point when nonempty, simply note that it does not contain any straight lines since $\mathbb{R}^n_+$ does not contain any straight lines.

(c) Let $n = |S|$ and let A be the $m \times n$ matrix whose columns are the elements of S. Let $b \in \operatorname{conv}(S)$. Consider the convex set $P = \{\lambda \in \mathbb{R}^n_+ : A\lambda = b, \sum_{i=1}^n \lambda_i = 1\}$. By part (b) we know that P has an extreme point with at most $m + 1$ nonzero components in λ. This is exactly what we want.

6 Extreme points in semidefinite programming

Part (a) of this exercise is the analogue of Exercise 5(a) for the case of semidefinite programming.

(a) Let $\mathcal{A} : \mathbf{S}^n \to \mathbb{R}^m$ be a linear map, $b \in \mathbb{R}^m$ and let $C = \{X \in \mathbf{S}^n_+ : \mathcal{A}(X) = b\}$. Show that any extreme point X of C satisfies $r(r+1)/2 \leq m$ where $r = \operatorname{rank} X$ [*Hint: Show that if X is an extreme point of C then* $\ker(\mathcal{A}) \cap \{Y \in \mathbf{S}^n : \operatorname{im}(Y) \subseteq \operatorname{im}(X)\} = \{0\}$].

Use Exercise 4 to show that if C is nonempty then it has at least one extreme point.

(b) Let $A, B \in \mathbf{S}^n$. Use part (a) to show that the set

$$R(A, B) = \{(x^T Ax, x^T Bx) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$$

is convex. (This set is known as the *numerical range* or *field of values* of the pair (A, B).) [*Hint: consider* $\{(\langle A, X\rangle, \langle B, X\rangle) : X \in \mathbf{S}^n_+\}$].

(c) Prove the following result, known as the *S-lemma*: Let $A, B \in \mathbf{S}^n$ and assume that for any $x \in \mathbb{R}^n$, $x^T A x \geq 0 \Rightarrow x^T B x \geq 0$. Assume furthermore that there exists $z \in \mathbb{R}^n$ such that $z^T A z > 0$. Show that there exists $\lambda \geq 0$ such that $B \succeq \lambda A$.

Give an example of $A, B \in \mathbf{S}^2$ to show that the condition of existence of $z \in \mathbb{R}^n$ such that $z^T A z > 0$ cannot be removed in general.

Solution.

(a) Let X be an extreme point of C. We will show that the only $Y \in \mathbf{S}^n$ that satisfies $\mathcal{A}(Y) = 0$ and $\mathrm{im}(Y) \subseteq \mathrm{im}(X)$ is $Y = 0$. Assume Y is such a point. Since X is positive definite on $\mathrm{im}(X)$ we have that $X \pm \epsilon Y \in C$ for small enough $\epsilon > 0$. Since X is an extreme point it must be that $Y = 0$.

Now observe that the set $\{Y \in \mathbf{S}^n : \mathrm{im}(Y) \subseteq \mathrm{im}(X)$ is a subspace of dimension $r(r+1)/2$ where $r = \dim \mathrm{im}(X) = \mathrm{rank}(X)$. Since the intersection of $\ker(\mathcal{A})$ and this subspace is $\{0\}$ it must be that $r(r+1)/2 \leq \mathrm{codim}_{\mathbf{S}^n}(\ker(\mathcal{A})) = m$.

Finally note that C does not contain any straight lines since $\mathbf{S}^n_+$ does not contain any straight lines. It thus follows from Exercise 4 that if C is not empty then it has at least one extreme point.

(b) Let $T = \{(\langle A, X \rangle, \langle B, X \rangle) : X \in \mathbf{S}^n_+\}$. It is clear that T is convex. We will show that $R(A, B) = T$. The inclusion $R(A, B) \subseteq T$ is easy: simply take $X = xx^T$. For the second inclusion let $(u, v) \in T$. Let $C = \{X \in \mathbf{S}^n_+ : \langle A, X \rangle = u, \ \langle B, X \rangle = v\}$. Since C is nonempty part (a) tells us that C contains at least one point X where $r = \mathrm{rank}(X)$ satisfies $r(r+1)/2 \leq 2$, and so $r = 1$. This means that $X = xx^T$ for some x and so $(u, v) = (x^T A x, x^T B x) \in R(A, B)$ as desired.

(c) The assumption tells us that the problem

$$\min_{x \in \mathbb{R}^n} x^T B x \quad : \quad x^T A x = 1 \tag{2}$$

is feasible and its optimal value is nonnegative. Consider the following semidefinite relaxation:

$$\min \quad \langle B, X \rangle \quad : \quad \langle A, X \rangle = 1, \ \ X \succeq 0. \tag{3}$$

By part (a) with $m = 2$ (see also argument of part (b)) we know that the optimal value of (2) and (3) are the same. The dual of the SDP (3) is:

$$\max \quad \lambda \quad : \quad B - \lambda A \succeq 0. \tag{4}$$

The assumption that there exists z such that $z^T A z > 0$ tells us that (3) is strictly feasible: indeed one can take $X = \epsilon I + \gamma z z^T$ with $\epsilon, \gamma > 0$ appropriately chosen such that $\langle A, X \rangle = 1$. By strong duality we know that the optimal value of (4) is also nonnegative. Thus this means there exists $\lambda \geq 0$ such that $B - \lambda A \succeq 0$.

To show that the assumption on the existence of z such that $z^T A z > 0$ is needed in general consider $A = \left[\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Note that $x^T A x \geq 0 \Rightarrow x^T B x \geq 0$ for any $x \in \mathbb{R}^2$. However $B - \lambda A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & \lambda \end{smallmatrix}\right]$ is not positive semidefinite for any choice of $\lambda \geq 0$.

7 Matrix square root

(a) Let $A, B \succ 0$. Show that if $A^2 \succeq B^2$ then $A \succeq B$ [*Hint: let v be an eigenvector of $A - B$ and consider* $v^T(A+B)(A-B)v$].

(b) Give an example of $A, B \in \mathbf{S}^2_{++}$ such that $A \succeq B$ but $A^2 \not\succeq B^2$.

Solution.

(a) Let λ be an eigenvalue of $A - B$ and v be an associated eigenvector. We want to show that $\lambda \geq 0$. On the one hand we have

$$v^T(A+B)(A-B)v = v^T(A^2 - B^2 + BA - AB)v = v^T(A^2 - B^2)v \geq 0$$

where we used the fact that $v^T(BA - AB)v = 0$ since $BA - AB$ is skew-symmetric. On the other hand we have

$$v^T(A+B)(A-B)v = \lambda v^T(A+B)v.$$

Since $v^T(A+B)v > 0$ (since $A + B \succ 0$) we get that $\lambda \geq 0$.

(b) Take $A = \begin{bmatrix} 5/2 & 0 \\ 0 & 4 \end{bmatrix} \succ 0$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \succ 0$. Then $A - B = \begin{bmatrix} 1/2 & -1 \\ -1 & 2 \end{bmatrix} \succeq 0$. However $A^2 - B^2 = \begin{bmatrix} 5/4 & -4 \\ -4 & 11 \end{bmatrix} \not\succeq 0$ because its determinant is $55/4 - 16 = -9/4 < 0$.

8 Newton polytope

For a polynomial $p(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \mathbf{x}^\alpha$ we define the Newton polytope of p to be

$$\text{Newton}(p) = \text{conv}\{\alpha \in \mathbb{N}^n : p_\alpha \neq 0\}.$$

(For example the Newton polytope of $p(\mathbf{x}) = x_1^3 x_2 + 2x_1x_2 - 4x_1x_2^2$ is $\text{conv}\{(3,1),(1,1),(1,2)\} \subset \mathbb{R}^2$.) Show that if

$$p = \sum_i q_i^2$$

then for all i, $\text{Newton}(q_i) \subseteq \frac{1}{2}\text{Newton}(p)$. [*Hint: consider an extreme point of* $\text{conv}(\bigcup_i \text{Newton}(q_i))$].

Solution. Consider $Q = \text{conv}\{\bigcup_i \text{Newton}(q_i)\}$. We will show that $Q \subseteq \frac{1}{2}\text{Newton}(p)$ and this will prove the claim. To do this let α be an extreme point of Q. We will show that necessarily $p_{2\alpha} > 0$. Note that the coefficient of $\mathbf{x}^{2\alpha}$ in $\sum_i q_i(\mathbf{x})^2$ is given by

$$\sum_i q_{i,\alpha}^2 + \sum_i \sum_{\substack{\gamma \neq \gamma' \\ \text{s.t.} \gamma + \gamma' = 2\alpha}} q_{i,\gamma} q_{i,\gamma'} \tag{5}$$

where $q_{i,\alpha}$ is the coefficient of the monomial $\mathbf{x}^\alpha$ in $q_i(\mathbf{x})$. Since α is an extreme point of Q there must exist at least one i such that $q_{i,\alpha} \neq 0$ and so the first term of (5) is positive. Also, by definition of extreme point, if $\gamma, \gamma' \in Q$ are such that $\frac{1}{2}(\gamma + \gamma') = \alpha$ then necessarily $\gamma = \gamma' = \alpha$. This shows that the second term of (5) is zero. This shows that the coefficient of $\mathbf{x}^{2\alpha}$ in $\sum_i q_i(\mathbf{x})^2 = p(\mathbf{x})$ is positive and so $2\alpha \in \text{Newton}(p)$. Since this is true for any extreme point α of Q, and Q is the convex hull of its extreme points, we get $Q \subseteq \frac{1}{2}\text{Newton}(p)$.

9 Homogeneous and nonhomogeneous polynomials

A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is called *homogeneous of degree d* if it only involves monomials of degree exactly d. Given a nonhomogeneous polynomial p of degree d we can *homogenise* it by introducing an additional variable x_0 via

$$\bar{p}(x_0, x_1, \ldots, x_n) = x_0^d p(x_1/x_0, \ldots, x_n/x_0) \tag{6}$$

(a) Show that (6) is well-defined. What is the homogenisation of $p(x_1, x_2) = x_1^2 x_2^2 - 2x_1x_2 + 1$?

(b) Show that p is nonnegative if and only if $\bar{p}$ is nonnegative.

(c) Show that p is a sum of squares if and only if $\bar{p}$ is a sum of squares.

(d) Show that if p is a homogeneous polynomial of degree $2d$ and $p = \sum_k q_k^2$ then the q_k must be homogeneous of degree d.

Solution.

(a) The operation (6) consists in replacing any monomial $c_\alpha \mathbf{x}^\alpha$ in p by $c_\alpha x_0^{d-|\alpha|} \mathbf{x}^\alpha$. The homogenisation of $p(x_1, x_2) = x_1^2 x_2^2 - 2x_1x_2 + 1$ is $p(x_0, x_1, x_2) = x_1^2 x_2^2 - 2x_0^2 x_1 x_2 + x_0^4$.

(b) If $\bar{p}$ is nonnegative then $p(x_1, \ldots, x_n) = \bar{p}(1, x_1, \ldots, x_n)$ is clearly nonnegative. Now assume $p \geq 0$ and let us show $\bar{p} \geq 0$. We know that $\deg p$ must be even. If $x_0 \neq 0$ then $\bar{p}(x_0, \ldots, x_n) \geq 0$ using (6). To show that $\bar{p}(x_0, x_1, \ldots, x_n) \geq 0$ when $x_0 = 0$ we can simply use a limit argument $\bar{p}(0, x_1, \ldots, x_n) = \lim_{x_0 \to 0} \bar{p}(x_0, \ldots, x_n)$.

(c) If $\bar{p}$ is a sum of squares then clearly p is also a sum of squares since $p(x_1, \ldots, x_n) = \bar{p}(1, x_1, \ldots, x_n)$. Conversely it is easy to verify that if $p = \sum_k q_k^2$ then $\bar{p} = \sum_k (\bar{q_k})^2$ where $\bar{q_k}$ are the homogenisation of q_k.

(d) Let $\mathbf{x}^\alpha$ be a monomial of smallest degree that has nonzero coefficient in any of the q_k. Then the coefficient of $\mathbf{x}^{2\alpha}$ in $p(\mathbf{x})$ must be strictly positive. Since p is homogeneous this means that $2|\alpha| = 2d$ i.e., $|\alpha| = d$. Similar argument shows that any monomial with nonzero coefficient in any of the q_k must have degree at most d. Thus all the monomials in any of the q_k must have degree d exactly.

10 A nonnegative polynomial that is not a sum of squares

In lecture we saw the Motzkin polynomial $M(x, y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ which is an explicit example of a nonnegative polynomial that is not a sum of squares in the case $(n, 2d) = (2, 6)$ (where n is the number of variables and $2d$ the degree). In this exercise we look at a polynomial in 3 variables of degree 4 (i.e., $(n, 2d) = (3, 4)$) that is nonnegative but not a sum-of-squares. Consider the following polynomial (due to Choi and Lam [CL77]).

$$Q(x, y, z) = x^2y^2 + x^2z^2 + y^2z^2 + 1 - 4xyz.$$

(a) Show that $Q(x, y, z) \geq 0$ for all $(x, y, z) \in \mathbb{R}^2$.

(b) Show that Q is not a sum of squares.

Solution.

(a) This follows directly from the arithmetic-geometric mean inequality.

(b) We give a proof similar to the one we saw in lecture concerning Motzkin polynomial. Assume $Q = \sum_k q_k^2$. Since Q has degree four we know that the q_k must be of degree 2. Write

$$q_k(x, y, z) = a_k x^2 + b_k y^2 + c_k z^2 + d_k xy + e_k xz + f_k yz + g_k x + h_k y + i_k z + j_k.$$

Since there are no terms x^4, y^4, z^4 in Q we get $a_k = b_k = c_k = 0$ for all k. Next since there are no terms x^2, y^2, z^2 in Q we get $g_k = h_k = i_k = 0$. But then there is no way to form the term $-4xyz$ in Q using $\sum_k q_k^2$.

11 Positive and decomposable maps

(Based on exercise 3.178 in [BPT12]) A map $\Lambda : \mathbf{S}^{n_1} \to \mathbf{S}^{n_2}$ is called *positive* if $\Lambda(A) \succeq 0$ whenever $A \succeq 0$.

(a) Show that if Λ has the form $\Lambda(A) = \sum_{i=1}^r P_i^T A P_i$ where $P_1, \dots, P_r \in \mathbb{R}^{n_1 \times n_2}$ then Λ is positive. Such maps are called *decomposable.*

(b) To any linear map $\Lambda : \mathbf{S}^{n_1} \to \mathbf{S}^{n_2}$ we can consider the polynomial $p(x, y) = y^T \Lambda(xx^T) y$ where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$. Show that Λ is a positive map if and only if p is nonnegative. Show that Λ is decomposable if and only if p is a sum-of-squares.

(c) Consider the following map $\Lambda : \mathbf{S}^3 \to \mathbf{S}^3$ due to M.-D. Choi [Cho75]:

$$\Lambda(A) = 2 \begin{bmatrix} a_{11} + a_{22} & 0 & 0 \\ 0 & a_{22} + a_{33} & 0 \\ 0 & 0 & a_{33} + a_{11} \end{bmatrix} - A.$$

(i) Show that Λ is positive [*Hint: in the case* $a_{33} \geq a_{11}$ *use* $\Lambda(A) = DAD + \begin{bmatrix} 2a_{22} & -2a_{12} & 0 \\ -2a_{12} & 2a_{33} & 0 \\ 0 & 0 & 2a_{11} \end{bmatrix}$ *with* $D = \operatorname{diag}(1, 1, -1)$*; then generalise using cyclic symmetry of* Λ].

(ii) Show that Λ is not decomposable. [*Hint: show that the associated polynomial* $p(x, y)$ *is not a sum-of-squares*].

Solution.

(a) If A is an $n_1 \times n_1$ positive semidefinite matrix and P is any $n_1 \times n_2$ matrix, then $P^T AP$ is positive semidefinite. Thus if A is positive semidefinite then $\sum_i P_i^T A P_i$ is also positive semidefinite and thus the map $A \mapsto \sum_i P_i^T A P_i$ is positive.

(b) We start by showing that Λ is positive if and only if p is nonnegative:

- Λ positive $\Rightarrow$ p nonnegative: If Λ is positive then $\Lambda(xx^T)$ is positive semidefinite for any $x \in \mathbb{R}^n$, and thus $y^T \Lambda(xx^T) y$ is nonnegative for all y. This shows that $p(x, y)$ is nonnegative.
- p nonnegative $\Rightarrow$ Λ positive: Assume that $p(x, y)$ is nonnegative. We will show that Λ is a positive map. Let $A = \sum_{i=1}^n x_i x_i^T$ be a positive semidefinite matrix. Then for any y we have $y^T \Lambda(A) y = \sum_{i=1}^n y^T \Lambda(x_i x_i^T) y = \sum_{i=1}^n p(x_i, y) \geq 0$. This is true for all y hence $\Lambda(A)$ is positive semidefinite.

We now show that Λ is decomposable if and only if p is a sum-of-squares.

- Λ decomposable $\Rightarrow$ p sum-of-squares: Assume that Λ is decomposable. We will show that $p(x,y)$ is a sum of squares. Let P_is be such that $\Lambda(A) = \sum_i P_i^T A P_i$. Then for any x and y we have: $p(x,y) = y^T(\sum_i P_i^T x x^T P_i)y = \sum_i (x^T P_i y)^T (x^T P_i y) = \sum_i q_i(x,y)^2$ where $q_i(x,y) = x^T P_i y$. Hence p is a sum of squares.
- p sum-of-squares $\Rightarrow$ Λ decomposable: Assume that p is a sum of squares. We will show that Λ is decomposable. Let q_is be such that $p(x,y) = \sum_i q_i(x,y)^2$. Observe that since p is homogeneous of degree 4, the q_i must be homogeneous of degree 2. Furthermore observe that q_i cannot contain monomials where the degree of an x_j or a y_j is greater than 1. In other words, this means that q_i has the form $q_i(x,y) = \sum_{k,l} (P_i)_{k,l} x_k y_l = x^T P_i y$. Hence for any x, y we have:

$$\begin{aligned} y^T \Lambda(xx^T) y = p(x,y) &= \sum_i q_i(x,y)^2 \\ &= \sum_i (x^T P_i y)^2 = \sum_i (x^T P_i y)^T (x^T P_i y) = y^T (\sum_i P_i^T x x^T P_i) y. \end{aligned}$$

  In other words we showed that for any fixed x, the following equality holds for any y: $y^T(\Lambda(xx^T) - (\sum_i P_i^T x x^T P_i))y = 0$ which implies $\Lambda(xx^T) - (\sum_i P_i^T x x^T P_i) = 0$ since the matrix is symmetric. Thus this shows that for any x we have $\Lambda(xx^T) = \sum_i P_i^T x x^T P_i$. Thus by linearity of Λ this shows that $\Lambda(A) = \sum_i P_i^T A P_i$ for any symmetric matrix A, and this means that Λ is decomposable. This completes the proof.

(c) (i) Let T be the cyclic permutation matrix $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and note that $\Lambda(TAT^T) = T\Lambda(A)T^T$. Let $A \succeq 0$ and note that after cyclically permuting the rows/columns we can assume $a_{33} \geq a_{11}$. Now observe that $\Lambda(A)$ can be written as:

$$\Lambda(A) = DAD + \begin{bmatrix} 2a_{22} & -2a_{12} & 0 \\ -2a_{12} & 2a_{33} & 0 \\ 0 & 0 & 2a_{11} \end{bmatrix}$$

where $D = \mathrm{diag}(1,1,-1)$. The first term is positive semidefinite. The second term also since $a_{11} \geq 0$ and the upper-left 2×2 has a determinant equal to $4(a_{22}a_{33} - a_{12}^2) \geq 4(a_{22}a_{11} - a_{12}^2) \geq 0$ since $A \succeq 0$.

(ii) We now show Λ is not decomposable by showing that the polynomial $p(x,y) = y^T \Lambda(xx^T) y$ is not a sum-of-squares. We have

$$\Lambda(xx^T) = \begin{bmatrix} x_1^2 + 2x_2^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_2^2 + 2x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_3^3 + 2x_1^2 \end{bmatrix}$$

The polynomial p in this case is

$$\begin{aligned} p(x,y) = x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 + 2(x_2^2y_1^2 + x_3^3y_2^2 + x_1^2y_3^2) \\ - 2(x_1x_2y_1y_2 + x_1x_3y_1y_3 + x_2x_3y_2y_3). \end{aligned}$$

Assume $p = \sum_k q_k^2$ where q_k are bilinear polynomials of the form

$$\begin{aligned} q_k(x,y) = a_kx_1y_1 + b_kx_1y_2 + c_kx_1y_3 \\ + d_kx_2y_1 + e_kx_2y_2 + f_kx_2y_3 \\ + g_kx_3y_1 + h_kx_3y_2 + i_kx_3y_3. \end{aligned}$$

Since p has no terms $x_1^2y_2^2, x_2^2y_3^2, x_3^2y_1^2$ we must get that $\sum_k b_k^2 = \sum_k f_k^2 = \sum_k g_k^2 = 0$ i.e., $b_k = f_k = g_k = 0$ for all k. Now considering the monomial $-2x_1x_2y_1y_2$ we get that $-2 = 2\sum_k a_ke_k$ i.e., $\sum_k a_ke_k = -1$. Similarly we get

$$\sum_k a_ke_k = \sum_k a_ki_k = \sum_k e_ki_k = -1. \tag{7}$$

On the other hand if we look at the monomials $x_1^2y_1^2$, $x_2^2y_2^2$, x_3^2, y_3^2 we get that

$$\sum_k a_k^2 = \sum_k e_k^2 = 1 = \sum_k i_k^2 = 1. \tag{8}$$

Combining (7) and (8) and using equality case for Cauchy-Schwarz we get a contradiction: indeed we must have $e_k = -a_k$ and $i_k = -a_k$ for all k but then $e_k = i_k$ which contradicts the last equality of (7).

References

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