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Mathematical Tripos Part II: Michaelmas Term 2021

Numerical Analysis – Lecture 14

Method 3.23 (The spectral method for evolutionary PDEs) We consider the problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \mathcal{L}u(x,t), & x \in [-1,1], \quad t \geq 0\,, \\ u(x,0) = g(x), & x \in [-1,1], \end{cases} \tag{3.20}$$

with appropriate boundary conditions on $\{-1,1\} \times \mathbb{R}_+$ and where $\mathcal{L}$ is a linear operator (acting on x), e.g., a differential operator. We want to solve this problem by the method of lines (semi-discretization), using a spectral method for the approximation of u and its derivatives in the spatial variable x. Then, in a general spectral method, we seek solutions $u_N(x,t)$ with

$$u_N(x,t) = \sum_{\#\{n\}=N} c_n(t)\, \varphi_n(x), \tag{3.21}$$

where $c_n(t)$ are expansion coefficients and φ_n are basis functions chosen according to the specific structure of (3.20). For example, we may take

1) the *Fourier expansion* with $c_n(t) = \widehat{u}_n(t)$, $\varphi_n(x) = e^{i\pi n x}$ for periodic boundary conditions,

2) a polynomial expansion such as the *Chebyshev expansion* with $c_n(t) = \breve{u}_n(t)$, $\varphi_n(x) = T_n(x)$ for other boundary conditions.

The spectral approximation in space (3.21) results into a $N \times N$ system of ODEs for the expansion coefficients $\{c_n(t)\}$:

$$\boldsymbol{c}' = B\boldsymbol{c}\,, \tag{3.22}$$

where $B \in \mathbb{R}^{N\times N}$, and $\boldsymbol{c} = \{c_n(t)\} \in \mathbb{R}^N$. We can solve it with standard ODE solvers (Euler, Crank-Nikolson, etc.) which as we have seen are approximations to the matrix exponent in the exact solution $\boldsymbol{c}(t) = \mathrm{e}^{tB}\boldsymbol{c}(0)$.

Example 3.24 (The diffusion equation) Consider the diffusion equation for a function $u = u(x,t)$,

$$\begin{cases} u_t = u_{xx}, & (x,t) \in [-1,1] \times \mathbb{R}_+\,, \\ u(x,0) = g(x), & x \in [-1,1]\,. \end{cases} \tag{3.23}$$

with the periodic boundary conditions $u(-1,t) = u(1,t)$, $u_x(-1,t) = u_x(1,t)$, and standard normalisation $\int_{-1}^{1} u(x,t)\,dx = 0$, both imposed for all values $t \geq 0$.

For each t, we approximate $u(x,t)$ by its N-th order partial Fourier sum in x,

$$u(x,t) \approx u_N(x,t) \ = \sum_{n \in \Gamma_N} \widehat{u}_n(t)\, e^{i\pi n x}\,, \qquad \Gamma_N := \{-N/2+1, ..., N/2\}\,.$$

Then, from (3.23), we see that each coefficient $\widehat{u}_n$ fulfills the ODE

$$\widehat{u}_n'(t) = -\pi^2 n^2 \widehat{u}_n(t)\,. \qquad n \in \Gamma_N \tag{3.24}$$

Its exact solution is $\widehat{u}_n(t) = \mathrm{e}^{-\pi^2 n^2 t}\,\widehat{g}_n$ for $n \neq 0$ and we set $\widehat{u}_0(t) = 0$ due to the normalisation condition, so that

$$u_N(x,t) = \sum_{n\in\Gamma_N} \widehat{g}_n\, \mathrm{e}^{-\pi^2 n^2 t}\, \mathrm{e}^{i\pi n x}\,,$$

which is the exact solution truncated to N terms.

Here, we were able to find the exact solution without solving ODE numerically due to the special structure of the Laplacian. However, for more general PDE we will need a numerical method, and thus the issue of stability arises, so we consider this issue on that simplified example.

Analysis 3.25 (Stability analysis) The system (3.24) has the form

$$\widehat{\boldsymbol{u}}' = B\widehat{\boldsymbol{u}}\,, \qquad B = \operatorname{diag}\{-\pi^2 n^2\}\,, \quad n \in \Gamma_N\,,$$

and we note that (a) all the eigenvalues of B are negative, and that (b) they consist of the eigenvalues $\lambda_n^{(2)}$ of the second order differentiation operator, with $\max|\lambda_n^{(2)}| = (\frac{N}{2})^2$.

If we approximate this system with the Euler method:

$$\widehat{\boldsymbol{u}}^{k+1} = (I + \tau B)\widehat{\boldsymbol{u}}^k, \qquad \tau := \Delta t,$$

then we see that, for stability condition $\|I+\tau B\| \le 1$, we need to scale teh time step $\tau = \Delta t \sim N^{-2}$.

Note that, for the Crank-Nikolson scheme, since the spectrum of B is negative, we get stability for any time step $\tau > 0$.

For general linear operator $\mathcal{L}$ in (3.20) with constant coefficients, the matix B is again diagonal (hence normal), and provided that it spectrum is negative, for stability we must scale the time step $\tau \sim N^{-m}$, where m is the maximal order of differentiation.

The scaling $\tau \sim N^{-2}$ may seem similar to the scaling $k \sim h^2$ in difference methods which we viewed as a disadvantage, however in spectral methods we can take N, the order of partial Fourier or Chebyshev sums to achieve a good appoximation, rather small. (We may still need to choose τ small enough to get a desired accuracy.)

Example 3.26 (The diffusion equation with non-constant coefficient) We want to solve the diffusion equation with a non-constant coefficient $a(x) > 0$ for a function $u = u(x,t)$

$$\begin{cases} u_t = (a(x)u_x)_x, & (x,t) \in [-1,1] \times \mathbb{R}_+\,, \\ u(x,0) = g(x), & x \in [-1,1]\,, \end{cases} \tag{3.25}$$

with boundary and normalization conditions as before. Approximating u by its partial Fourier sum results in the following system of ODEs for the coefficients $\widehat{u}_n$

$$\widehat{u}_n'(t) = -\pi^2 \sum_{m\in\Gamma_N} mn\,\widehat{a}_{n-m}\,\widehat{u}_m(t), \qquad n \in \Gamma_N\,.$$

For the discretization in time we may apply the Euler method, this gives

$$\widehat{u}_n^{k+1} = \widehat{u}_n^k - \tau\,\pi^2 \sum_{m\in\Gamma_N} mn\,\widehat{a}_{n-m}\,\widehat{u}_m^k\,, \qquad \tau = \Delta t\,,$$

or in the vector form

$$\widehat{\boldsymbol{u}}^{k+1} = (I + \tau B)\widehat{\boldsymbol{u}}^k,$$

where $B = (b_{m,n}) = (-\pi^2 mn\,\widehat{a}_{n-m})$. For stability of Euler method, we again need $\|I + \tau B\| \le 1$, but analysis here is less straightforward.

Matlab demo: See the online documentation *Using Chebyshev Spectral Methods* at `http://www.damtp.cam.ac.uk/user/hf323/M21-II-NA/demos/chebyshev/chebyshev.html` for a simple example of how boundary conditions can be installed.