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Mathematical Tripos Part II: Michaelmas Term 2022

Numerical Analysis – Lecture 12

The algebra of Fourier expansions Let $\mathcal{A}$ be the set of all 2-periodic functions f which are analytic on a horizontal strip $\{z \in \mathbb{C} : -a < \operatorname{Im} z < a\}$. Then $\mathcal{A}$ is a linear space, i.e., $f, g \in \mathcal{A}$ and $\alpha \in \mathbb{C}$ then $f + g \in \mathcal{A}$ and $\alpha f \in \mathcal{A}$. In particular, with f and g expressed in its Fourier series, i.e.,

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}_n e^{i\pi nx}, \quad g(x) = \sum_{n=-\infty}^{\infty} \widehat{g}_n e^{i\pi nx}$$

we have

$$f(x) + g(x) = \sum_{n=-\infty}^{\infty} (\widehat{f}_n + \widehat{g}_n) e^{i\pi nx}, \quad \alpha f(x) = \sum_{n=-\infty}^{\infty} \alpha \widehat{f}_n e^{i\pi nx} \tag{3.3}$$

and

$$f(x) \cdot g(x) = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} \widehat{f}_{n-m} \widehat{g}_m \right) e^{i\pi nx} = \sum_{n=-\infty}^{\infty} \left(\widehat{f} * \widehat{g}\right)_n e^{i\pi nx}, \tag{3.4}$$

where $*$ denotes the convolution operator, hence $\widehat{(f \cdot g)}_n = (\widehat{f} * \widehat{g})_n$. Moreover, if $f \in \mathcal{A}$ then $f' \in \mathcal{A}$ and

$$f'(x) = i\pi \sum_{n=-\infty}^{\infty} n \cdot \widehat{f}_n e^{i\pi nx}. \tag{3.5}$$

Since $\{\widehat{f}_n\}$ decays exponentially fast, this shows that all derivatives of f have rapidly convergent Fourier expansions.

Example 3.8 (Application to differential equations) Consider the two-point boundary value problem: $y = y(x)$, $-1 \le x \le 1$, solves

$$y'' + a(x) y' + b(x) y = f(x), \quad y(-1) = y(1), \tag{3.6}$$

where $a, b, f \in \mathcal{A}$ and we seek a *periodic solution* $y \in \mathcal{A}$ for (3.6). Substituting y, a, b and f by their Fourier series and using (3.3)-(3.5) we obtain an infinite dimensional system of linear equations for the Fourier coefficients $\widehat{y}_n$:

$$-\pi^2 n^2 \widehat{y}_n + i\pi \sum_{m=-\infty}^{\infty} m \widehat{a}_{n-m} \widehat{y}_m + \sum_{m=-\infty}^{\infty} \widehat{b}_{n-m} \widehat{y}_m = \widehat{f}_n, \quad n \in \mathbb{Z}. \tag{3.7}$$

Since $a, b, f \in \mathcal{A}$, their Fourier coefficients decrease exponentially fast. Hence, we can truncate (3.7) into the N-dimensional system

$$-\pi^2 n^2 \widehat{y}_n + i\pi \sum_{m=-N/2+1}^{N/2} m \widehat{a}_{n-m} \widehat{y}_m + \sum_{m=-N/2+1}^{N/2} \widehat{b}_{n-m} \widehat{y}_m = \widehat{f}_n, \qquad n = -N/2+1, \dots, N/2. \tag{3.8}$$

Remark 3.9 The matrix of (3.8) is in general dense, but our theory predicts that fairly small values of N, hence very small matrices, are sufficient for high accuracy. For instance: choosing $a(x) = f(x) = \cos \pi x$, $b(x) = \sin 2\pi x$ (which incidentally even leads to a sparse matrix) we get

$N = 16$	error of size 10^{-10}
$N = 22$	error of size 10^{-15} (which is already hitting the accuracy of computer arithmetic)

Computation of Fourier coefficients (DFT) To form the linear system (3.8), we need to compute the Fourier coefficients of $a(x), b(x)$, and $f(x)$, i.e., we need to compute integrals of the form:

$$\widehat{f}_n = \frac{1}{2}\int_{-1}^{1} f(t)e^{-i\pi nt}\,dt, \quad n \in \mathbb{Z}. \tag{3.9}$$

Call $h(t) = f(t)e^{-i\pi nt}$. If $f \in \mathcal{A}$, then so is h. One simple way to approximate the integral of h on $[-1, 1]$ is using the *rectangle rule*:

$$\int_{-1}^{1} h(t)\,dt \approx \frac{2}{N}\sum_{k=-N/2+1}^{N/2} h\left(\frac{2k}{N}\right). \tag{3.10}$$

This approximation happens to be exponentially convergent in N.

Theorem 3.10 *Let h be a 2-periodic function such that its Fourier series is absolutely convergent. Let $I(h) = \int_{-1}^{1} h(t)dt$, and for an even integer N, let $I_N(h) = \frac{2}{N}\sum_{k=-N/2+1}^{N/2} h\left(\frac{2k}{N}\right)$. Then*

$$I_N(h) - I(h) = 2\sum_{r\in\mathbb{Z},|r|\geq 1} \hat{h}_{Nr}. \tag{3.11}$$

As a consequence, if h is analytic on the horizontal strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < a\}$ and $|h(z)| \leq M$ for $|\operatorname{Im} z| < a$, then by letting $c = e^{-a\pi} \in (0, 1)$, we have $|I_N(h) - I(h)| \leq 4Mc^N/(1 - c^N)$.

Remark 3.11 *Another consequence of the expression* (3.11) *is that $I_N(h) = I(h)$ if h is a trigonometric polynomial of degree $< N$, i.e., if $\hat{h}_n = 0$ for $|n| \geq N$. This is reminiscent of Gaussian quadrature rules which are exact for polynomials up to degree $2N - 1$. For more on the exponential convergence of the rectangle rule for periodic analytic functions, we refer the interested reader to the following review article The Exponentially Convergent Trapezoidal Rule, SIAM Review, 2014 by L. N. Trefethen, and J. A. C. Weideman.*

Proof. Let $\omega_N = e^{2\pi i/N}$. Then we have

$$\frac{2}{N}\sum_{k=-N/2+1}^{N/2} h\left(\frac{2k}{N}\right) = \frac{2}{N}\sum_{k=-N/2+1}^{N/2}\sum_{n=-\infty}^{\infty} \widehat{h}_n e^{2\pi ink/N} = \frac{2}{N}\sum_{n=-\infty}^{\infty}\widehat{h}_n \sum_{k=-N/2+1}^{N/2} \omega_N^{nk}.$$

Since $\omega_N^N = 1$ we have

$$\sum_{k=-N/2+1}^{N/2} \omega_N^{nk} = \omega_N^{-n(N/2-1)}\sum_{k=0}^{N-1}\omega_N^{nk} = \begin{cases} N, & n \equiv 0 \pmod N, \\ 0, & n \not\equiv 0 \pmod N, \end{cases}$$

and we deduce that

$$\frac{2}{N}\sum_{k=-N/2+1}^{N/2} h\left(\frac{2k}{N}\right) = 2\sum_{r=-\infty}^{\infty}\widehat{h}_{Nr}.$$

Since $I(h) = 2\hat{h}_0$, we immediately obtain the expression (3.11).

For the second part of the theorem, the analyticity assumption guarantees, that the Fourier coefficients $|\hat{h}_n|$ decay exponentially fast, namely $|\hat{h}_n| \leq Mc^{|N|}$ (see Lecture 11). In this case we have

$$2\sum_{r\in\mathbb{Z},|r|\geq 1} |\hat{h}_{Nr}| \leq 4M\sum_{r=1}^{\infty} c^{Nr} = 4Mc^N/(1 - c^N)$$

as desired.

Remark 3.12 Applying the rectangle rule to the integral in (3.9) corresponds to the approximation

$$\hat{f}_n \approx \frac{1}{N} \sum_{k=-N/2+1}^{N/2} f\left(\frac{2k}{N}\right) e^{-2ik\pi n/N}.$$

We recognize that the right-hand side, for $n = -N/2+1, \ldots, N/2$, corresponds to the discrete Fourier transform of the sequence $(y_k) = \left(f(\frac{2k}{N})\right)$. Thus, one can compute the approximations to $\hat{f}_n$ using the FFT algorithm.

Problem 3.13 (The Poisson equation) We consider the *Poisson equation*

$$\nabla^2 u = f, \quad -1 \leq x, y \leq 1, \tag{3.12}$$

where f is analytic and obeys the periodic boundary conditions

$$f(-1,y) = f(1,y), \quad -1 \leq y \leq 1, \qquad f(x,-1) = f(x,1), \quad -1 \leq x \leq 1.$$

Moreover, we add to (3.12) the following *periodic boundary conditions*

$$\begin{aligned} &u(-1,y) = u(1,y), \quad u_x(-1,y) = u_x(1,y), \quad -1 \leq y \leq 1 \\ &u(x,-1) = u(x,1), \quad u_y(x,-1) = u_y(x,1), \quad -1 \leq x \leq 1. \end{aligned} \tag{3.13}$$

With these boundary conditions alone, a solution of (3.12) is only defined up to an additive constant. Hence, we add a *normalisation condition* to fix the constant:

$$\int_{-1}^{1} \int_{-1}^{1} u(x,y)\, dx\, dy = 0. \tag{3.14}$$

We have the spectrally convergent Fourier expansion

$$f(x,y) = \sum_{k,l=-\infty}^{\infty} \widehat{f}_{k,l} e^{i\pi(kx+ly)}$$

and seek the Fourier expansion of u

$$u(x,y) = \sum_{k,l=-\infty}^{\infty} \widehat{u}_{k,l} e^{i\pi(kx+ly)}.$$

Since

$$0 = \int_{-1}^{1} \int_{-1}^{1} u(x,y)\, dx\, dy = \sum_{k,l=-\infty}^{\infty} \widehat{u}_{k,l} \int_{-1}^{1} \int_{-1}^{1} e^{i\pi(kx+ly)}\, dx\, dy = \widehat{u}_{0,0},$$

and

$$\nabla^2 u(x,y) = -\pi^2 \sum_{k,l=-\infty}^{\infty} (k^2+l^2) \widehat{u}_{k,l} e^{i\pi(kx+ly)},$$

together with (3.12), we have

$$\begin{cases} \widehat{u}_{k,l} = -\dfrac{1}{(k^2+l^2)\pi^2} \widehat{f}_{k,l}, \quad k,l \in \mathbb{Z},\ (k,l) \neq (0,0) \\ \widehat{u}_{0,0} = 0. \end{cases}$$

Remark 3.14 Applying a spectral method to the Poisson equation is not representative for its application to other PDEs. The reason is the special structure of the Poisson equation. In fact, $\phi_{k,l} = e^{i\pi(kx+ly)}$ are the eigenfunctions of the Laplace operator with

$$\nabla^2 \phi_{k,l} = -\pi^2 (k^2+l^2) \phi_{k,l},$$

and they obey periodic boundary conditions.

Problem 3.15 (General second-order linear elliptic PDE) We consider the more general second-order linear elliptic PDE

$$\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a(x,y)\frac{\partial u}{\partial y}\right)=f, \quad -1\leq x,y\leq 1,$$

with $a(x,y)>0$, and a and f periodic. We again impose the periodic boundary conditions (3.13) and the normalisation condition (3.14). We use the Fourier expansions

$$g(x,y)=\sum_{k,l\in\mathbb{Z}}\widehat{g}_{k,l}e^{i\pi(kx+ly)}, \qquad h(x,y)=\sum_{m,n\in\mathbb{Z}}\widehat{h}_{m,n}e^{i\pi(mx+ny)},$$

together with the bivariate versions of (3.4)-(3.5)

$$\widehat{(g\cdot h)}_{k,l}=\sum_{m,n\in\mathbb{Z}}\widehat{g}_{k-m,l-n}\widehat{h}_{m,n}, \qquad \widehat{(g_x)}_{k,l}=i\pi k\,\widehat{g}_{k,l}\,, \qquad \widehat{(g_y)}_{k,l}=i\pi l\,\widehat{g}_{k,l}\,,$$

$$\widehat{(h_x)}_{m,n}=i\pi m\,\widehat{h}_{m,n}\,, \qquad \widehat{(h_y)}_{m,n}=i\pi n\,\widehat{h}_{m,n}\,.$$

This gives

$$-\pi^2\sum_{k,l\in\mathbb{Z}}\sum_{m,n\in\mathbb{Z}}(km+ln)\,\widehat{a}_{k-m,l-n}\widehat{u}_{m,n}e^{i\pi(kx+ly)}=\sum_{k,l\in\mathbb{Z}}\widehat{f}_{k,l}e^{i\pi(kx+ly)}\,.$$

In the next steps, we truncate the expansions to $-N/2+1\leq k,l,m,n\leq N/2$ and impose the normalisation condition $\widehat{u}_{0,0}=0$. This results in a system of N^2-1 linear algebraic equations in the unknowns $\widehat{u}_{m,n}$, where $m,n=-N/2+1...N/2$, and $(m,n)\neq(0,0)$:

$$\sum_{m,n=-N/2+1}^{N/2}(km+ln)\,\widehat{a}_{k-m,l-n}\,\widehat{u}_{m,n}=-\frac{1}{\pi^2}\,\widehat{f}_{k,l}\,, \qquad k,l=-N/2+1...N/2\,.$$

Discussion 3.16 (Analyticity and periodicity) The fast convergence of spectral methods rests on two properties of the underlying problem: analyticity and periodicity. If one is not satisfied the rate of convergence in general drops to polynomial. However, to a certain extent, we can relax these two assumptions while still retaining the substantive advantages of Fourier expansions.

- *Relaxing analyticity*: In general, the speed of convergence of the truncated Fourier series of a function f depends on the smoothness of the function. In fact, the smoother the function the faster the truncated series converges, i.e., for $f\in C^p(-1,1)$ we receive an $\mathcal{O}(N^{-p})$ order of convergence.

- *Relaxing periodicity*: Disappointingly, periodicity is necessary for spectral convergence. One way around this is to change our set of basis functions, e.g., to Chebyshev polynomials.