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Mathematical Tripos Part II: Michaelmas Term 2022

Numerical Analysis – Lecture 13

Chebyshev polynomials The Chebyshev polynomial of degree n is defined as

$$T_n(x) := \cos(n \arccos x), \quad x \in [-1, 1], \tag{3.14}$$

or equivalently, by the identity $T_n(\cos\theta) = \cos(n\theta)$ for $\theta \in [0, 2\pi]$.

1) The sequence (T_n) obeys the three-term recurrence relation

$$\begin{aligned} &T_0(x) \equiv 1, \quad T_1(x) = x, \\ &T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \end{aligned}$$

in particular, T_n is indeed an algebraic polynomial of degree n, with the leading coefficient 2^{n-1}. (The recurrence is due to the equality $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$ via substitution $x = \cos\theta$, expressions for T_0 and T_1 are straightforward.)

2) Also, (T_n) forms a sequence of orthogonal polynomials with respect to the inner product $(f, g)_w := \int_{-1}^{1} f(x)g(x)w(x)dx$, with the weight function $w(x) := (1 - x^2)^{-1/2}$. Namely, we have

$$(T_n, T_m)_w = \int_{-1}^{1} T_m(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}} = \int_0^\pi \cos m\theta \cos n\theta\, d\theta = \begin{cases} \pi, & m = n = 0\,, \\ \frac{\pi}{2}, & m = n \geq 1\,, \\ 0, & m \neq n\,. \end{cases} \tag{3.15}$$

Chebyshev expansion Since $(T_n)_{n=0}^\infty$ forms an orthogonal sequence, a function f such that $\int_{-1}^{1} |f(x)|^2 w(x)\,dx < \infty$ can be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} \breve{f}_n T_n(x),$$

with the Chebyshev coefficients $\breve{f}_n$. Making inner product of both sides with T_n and using orthogonality yields

$$(f, T_n)_w = \breve{f}_n (T_n, T_n)_w \quad \Rightarrow \quad \breve{f}_n = \frac{(f, T_n)_w}{(T_n, T_n)_w} = \frac{c_n}{\pi} \int_{-1}^{1} f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}}\,, \tag{3.16}$$

where $c_0 = 1$ and $c_n = 2$ for $n \geq 1$.

Connection to the Fourier expansion. Letting $x = \cos t\pi$ and $g(t) = f(\cos(t\pi))$, we obtain

$$\int_{-1}^{1} f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}} = \pi \int_0^1 f(\cos t\pi)T_n(\cos t\pi)\,dt = \frac{\pi}{2}\int_{-1}^{1} g(t)\cos nt\pi\,dt\,. \tag{3.17}$$

Given that $\cos nt\pi = \frac{1}{2}(e^{int\pi} + e^{-int\pi})$, and using the Fourier expansion of the 2-periodic function g,

$$g(t) = \sum_{n\in\mathbb{Z}} \widehat{g}_n e^{in\pi t}, \quad \text{where} \quad \widehat{g}_n = \frac{1}{2}\int_{-1}^{1} g(t)e^{-int\pi}\,dt, \quad n \in \mathbb{Z}\,,$$

we continue (3.17) as

$$\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}(\widehat{g}_{-n} + \widehat{g}_n)\,,$$

and from (3.16) we deduce that

$$\breve{f}_n = \begin{cases} \widehat{g}_0, & n = 0\,, \\ \widehat{g}_{-n} + \widehat{g}_n = 2\widehat{g}_n, & n \geq 1\,. \end{cases} \tag{3.18}$$

Discussion 3.17 (Properties of the Chebyshev expansion) As we have seen, for a general integrable function f, the computation of its Chebyshev expansion is equivalent to the Fourier expansion of the function $g(\theta) = f(\cos\theta)$. Since the latter is periodic with period 2π, we can use a discrete Fourier transform (DFT) to compute the Chebyshev coefficients $\breve{f}_n$. [Actually, based on this connection, one can perform a direct fast Chebyshev transform].

Also, if f can be analytically extended from $[-1, 1]$ (to the so-called Bernstein ellipse), then $\breve{f}_n$ decays spectrally fast for $n \gg 1$ (with the rate depending on the size of the ellipse). Hence, the Chebyshev expansion inherits the rapid convergence of spectral methods without assuming that f is periodic.

Theorem 3.18 *Let f be a function on $[-1, 1]$ such that it can be extended analytically to the* Bernstein ellipse *in the complex plane*

$$B(a) = \left\{ z = x + iy \in \mathbb{C} : \frac{x^2}{\cosh^2(a\pi)} + \frac{y^2}{\sinh^2(a\pi)} < 1 \right\} \tag{3.19}$$

where $a > 0$, and assume furthermore that $|f(z)| \leq M$ for $z \in B(a)$. Then with $c = e^{-a\pi} \in (0, 1)$, we have $|\breve{f}_n| \leq 2Mc^n$ for $n \geq 1$, and $|f(x) - \sum_{n=0}^{N-1} \breve{f}_n T_n(x)| \leq 2Mc^N/(1-c)$.

Proof. Let $g(t) = f(\cos(t\pi)) = f((e^{it\pi} + e^{-it\pi})/2)$ which is 2-periodic. Let $S(a) = \{z \in \mathbb{C} : |\operatorname{Im} z| < a\}$, and note that

$$t \in S(a) \iff \cos(t\pi) \in B(a). \tag{3.20}$$

(See below for justification.) Since f is assumed analytic on $B(a)$, it follows that g is analytic on $S(a)$. From the theorem of Lecture 11, we know that $|\hat{g}_n| \leq Me^{-a\pi|n|}$, and thus by (3.18), it follows that $|\breve{f}_0| \leq M$ and $|\breve{f}_n| \leq 2Me^{-a\pi n}$ for $n \geq 1$. Furthermore, we have, for any $x \in [-1, 1]$

$$|f(x) - \sum_{n=0}^{N-1} \breve{f}_n T_n(x)| \leq \sum_{n=N}^{\infty} |\breve{f}_n||T_n(x)| \leq \sum_{n=N}^{\infty} |\breve{f}_n| \leq 2Mc^N/(1-c)$$

as desired.

It remains to prove (3.20). For $b > 0$ and $x \in \mathbb{R}$, we have

$$\cos(x + ib) = \tfrac{1}{2}(e^{i(x+ib)} + e^{-i(x+ib)}) = \tfrac{1}{2}(e^{-b}e^{ix} + e^{b}e^{-ix})$$

and thus $\operatorname{Re}(\cos(x + ib)) = \cosh(b)\cos(x)$ and $\operatorname{Im}(\cos(x + ib)) = -\sinh(b)\sin(x)$. This shows that $\{\cos(x + ib) : x \in \mathbb{R}\}$ is precisely the ellipse of equation $x^2/\cosh(b)^2 + y^2/\sinh(b)^2 = 1$. $\square$

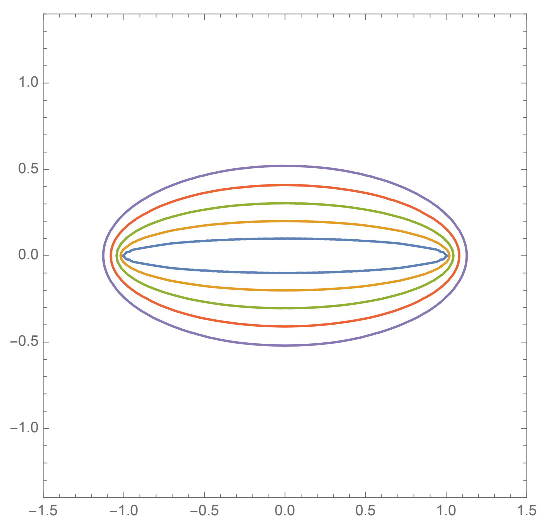

Figure 1: Bernstein ellipses $B(a)$ as defined in (3.19) or different values of $a > 0$.

The algebra of Chebyshev expansions In order to use spectral Galerkin methods with the Chebyshev basis, we need to understand how Chebyshev expansion behaves under pointwise multiplication of functions, and differentiation. Starting by the multiplication operation, we see that

$$\begin{aligned} T_m(x)T_n(x) &= \cos(m\theta)\cos(n\theta) \\ &= \tfrac{1}{2}\left[\cos((m-n)\theta)+\cos((m+n)\theta)\right] \\ &= \tfrac{1}{2}\left[T_{|m-n|}(x)+T_{m+n}(x)\right] \end{aligned}$$

and hence,

$$\begin{aligned} f(x)g(x) &= \sum_{m=0}^{\infty}\breve{f}_m T_m(x)\cdot\sum_{n=0}^{\infty}\breve{g}_n T_n(x) = \frac{1}{2}\sum_{m,n=0}^{\infty}\breve{f}_m\breve{g}_n\left[T_{|m-n|}(x)+T_{m+n}(x)\right] \\ &= \frac{1}{2}\sum_{k=0}^{\infty}T_k(x)\left(\sum_{\substack{m,n\geq 0\\ m+n=k}}\breve{f}_m\breve{g}_n+\sum_{\substack{m,n\geq 0\\ |m-n|=k}}\breve{f}_m\breve{g}_n\right). \end{aligned}$$

Lemma 3.19 (Derivatives of Chebyshev polynomials) *We can express derivatives T_n' in terms of (T_k) as follows,*

$$T_{2n}'(x) = (2n)\cdot 2\sum_{k=1}^{n}T_{2k-1}(x), \tag{3.21}$$

$$T_{2n+1}'(x) = (2n+1)\left[T_0(x)+2\sum_{k=1}^{n}T_{2k}(x)\right]. \tag{3.22}$$

Proof. From (3.14), we deduce

$$T_m(x)=\cos m\theta \quad\Rightarrow\quad T_m'(x)=\frac{m\sin m\theta}{\sin\theta} \qquad x=\cos\theta\,.$$

So, for $m=2n$, (3.21) follows from the identity $\frac{\sin 2n\theta}{\sin\theta}=2\sum_{k=1}^{n}\cos(2k-1)\theta$, which is verified as

$$2\sin\theta\sum_{k=1}^{n}\cos(2k-1)\theta=\sum_{k=1}^{n}2\cos(2k-1)\theta\sin\theta=\sum_{k=1}^{n}\left[\sin 2k\theta-\sin 2(k-1)\theta\right]=\sin 2n\theta.$$

For $m=2n+1$, (3.22) turns into identity $\frac{\sin(2n+1)\theta}{\sin\theta}=1+2\sum_{k=1}^{n}\cos 2k\theta$, and that follows from

$$\sin\theta\cdot\Big(1+2\sum_{k=1}^{n}\cos 2k\theta\Big)=\sin\theta+\sum_{k=1}^{n}\left[\sin(2k+1)\theta-\sin(2k-1)\theta\right]=\sin(2n+1)\theta.$$

□

The lemma above allows us to express the Chebyshev coefficients of the derivative of a function f, in terms of those of f. We get

$$\begin{cases} \breve{f'}_0 &= \breve{f}_1+3\breve{f}_3+5\breve{f}_5+\cdots \\ \breve{f'}_1 &= 2(2\breve{f}_2+4\breve{f}_4+6\breve{f}_6+\cdots) \\ \breve{f'}_2 &= 2(3\breve{f}_3+5\breve{f}_5+\cdots) \\ \breve{f'}_3 &= 2(4\breve{f}_4+6\breve{f}_6+\cdots) \\ \vdots \end{cases}$$

In general, for the k'th derivative we get:

$$\widetilde{f^{(k)}}_n = c_n\sum_{\substack{m=n+1\\ n+m\text{ odd}}}^{\infty} m\ \widetilde{f^{(k-1)}}_m,\quad \forall\, k\geq 1,$$

where $c_0=1$ and $c_n=2$ for $n\geq 1$.