Dr H. Fawzi

Mathematical Tripos Part II: Michaelmas Term 2023

Numerical Analysis – Lecture 4

Fast Fourier Transform (FFT) We assume that n is a power of 2, i.e. $n = 2m = 2^p$, and for $\boldsymbol{y} \in \mathbb{C}^{2m}$, denote by

$$\boldsymbol{y}^{(\mathrm{E})} = \{y_{2j}\}_{j=0,\ldots,m} \in \mathbb{C}^m \qquad \text{and} \qquad \boldsymbol{y}^{(\mathrm{O})} = \{y_{2j+1}\}_{j=0,\ldots,m} \in \mathbb{C}^m$$

the even and odd portions of $\boldsymbol{y}$, respectively.

Suppose that we already know the DFT of both 'short' sequences,

$$\boldsymbol{x}^{(\mathrm{E})} = \mathcal{F}_m \boldsymbol{y}^{(\mathrm{E})}, \qquad \boldsymbol{x}^{(\mathrm{O})} = \mathcal{F}_m \boldsymbol{y}^{(\mathrm{O})}.$$

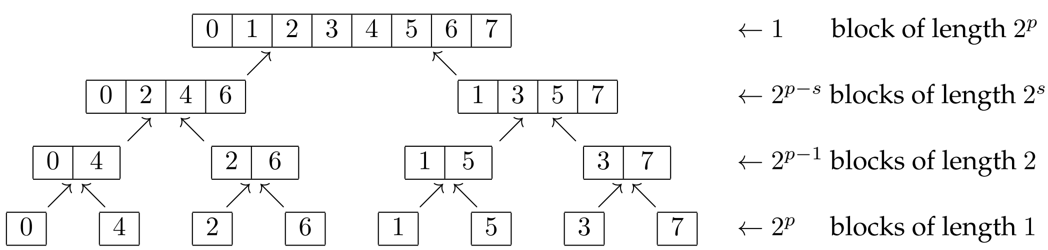
It is then possible to assemble $\boldsymbol{x} = \mathcal{F}_{2m}\boldsymbol{y}$ in a small number of operations. Indeed, for $\ell \in \{0, \ldots, m-1\}$, we have

$$\begin{aligned} x_\ell = \sum_{j=0}^{2m-1} \omega_{2m}^{j\ell} y_j &= \sum_{j=0}^{m-1} \omega_{2m}^{2j\ell} y_{2j} + \sum_{j=0}^{m-1} \omega_{2m}^{(2j+1)\ell} y_{2j+1} \\ &= \sum_{j=0}^{m-1} \omega_m^{j\ell} y_j^{(\mathrm{E})} + \omega_{2m}^{\ell} \sum_{j=0}^{m-1} \omega_m^{j\ell} y_j^{(\mathrm{O})} = x_\ell^{(\mathrm{E})} + \omega_{2m}^\ell x_\ell^{(\mathrm{O})}. \end{aligned}$$

Therefore, it costs just m products to evaluate the first half of $\boldsymbol{x}$, provided that $\boldsymbol{x}^{(\mathrm{E})}$ and $\boldsymbol{x}^{(\mathrm{O})}$ are known. It actually costs nothing to evaluate the second half, since

$$\omega_m^{j(m+\ell)} = \omega_m^{j\ell}, \qquad \omega_{2m}^{m+\ell} = -\omega_{2m}^{\ell} \qquad \Rightarrow \qquad x_{m+\ell} = x_\ell^{(\mathrm{E})} - \omega_{2m}^\ell x_\ell^{(\mathrm{O})}, \qquad \ell = 0, \ldots, m-1.$$

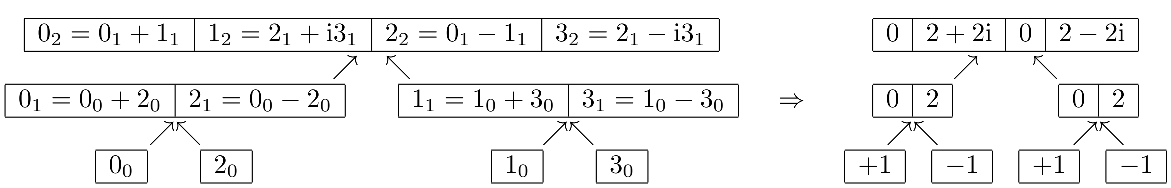
To execute FFT, we start from vectors of unit length and in each s-th stage, $s = 1 \ldots p$, assemble 2^{p-s} vectors of length 2^s from vectors of length 2^{s-1}: this costs $2^{p-s}2^{s-1} = 2^{p-1}$ products. Altogether, the cost of FFT is $p2^{p-1} = \frac{1}{2} n \log_2 n$ products.



For $n = 1024 = 2^{10}$, say, the cost is $\approx 5 \times 10^3$ products, compared to $\approx 10^6$ for naive matrix multiplication! For $n = 2^{20}$ the respective numbers are $\approx 1.05 \times 10^7$ and $\approx 1.1 \times 10^{12}$, which represents a saving by a factor of more than 10^5.

Matlab demo: Check out the online animation for computing the FFT at `http://www.damtp.cam.ac.uk/user/hf323/M21-II-NA/demos/fft_gui/fft_gui.html` and download the Matlab GUI from there to follow the computation of each single FFT term.

Example 1.15 Computation of FFT for $n = 4$ in general, and for the vector $\boldsymbol{y} = (1, 1, -1, -1)$ in particular.



2 Partial differential equations of evolution

We consider the *diffusion equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad 0 \le x \le 1, \quad t \ge 0,$$

with *initial conditions* $u(x,0) = u_0(x)$ for $t = 0$ and zero Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$. By Taylor's expansion

$$\begin{aligned}
\tfrac{\partial u(x,t)}{\partial t} &= \tfrac{1}{k}\big[u(x,t+k) - u(x,t)\big] + \mathcal{O}(k), && k = \Delta t\,,\\
\tfrac{\partial^2 u(x,t)}{\partial x^2} &= \tfrac{1}{h^2}\big[u(x-h,t) - 2u(x,t) + u(x+h,t)\big] + \mathcal{O}(h^2), && h = \Delta x\,,
\end{aligned}$$

so that, for the exact solution $u = \widehat{u}$ of the diffusion equation, we obtain

$$\widehat{u}(x,t+k) = \widehat{u}(x,t) + \tfrac{k}{h^2}\big[\widehat{u}(x-h,t) - 2\widehat{u}(x,t) + \widehat{u}(x+h,t)\big] + \eta(x,t) \tag{2.1}$$

where $\eta(x,t) = \mathcal{O}(k^2 + kh^2)$. (More precisely, one proves using Taylor's theorem that $|\eta(x,t)| \le c_1 k^2 + c_2 k h^2$ where $c_1 = \frac{1}{2}\max_{\xi,\tau}|\frac{\partial^2 \widehat{u}}{\partial t^2}(\xi,\tau)|$ and $c_2 = \frac{1}{12}\max_{\xi,\tau}|\frac{\partial^4 \widehat{u}}{\partial x^4}(\xi,\tau)|$.) That motivates the numerical scheme for approximation $u_m^n \approx \widehat{u}(x_m, t_n)$ on the rectangular mesh $(x_m, t_n) = (mh, nk)$:

$$u_m^{n+1} = u_m^n + \mu\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right), \qquad m = 1...M\,. \tag{2.2}$$

Here $h = \frac{1}{M+1}$ and $\mu = \frac{k}{h^2} = \frac{\Delta t}{(\Delta x)^2}$ is the so-called *Courant number.* With μ being fixed, we have $k = \mu h^2$, so that the local truncation error of the scheme is $\mathcal{O}(k^2)$. Substituting whenever necessary initial conditions u_m^0 and boundary conditions u_0^n and u_{M+1}^n, we possess enough information to advance in (2.2) from $\boldsymbol{u}^n := [u_1^n, \ldots, u_M^n]$ to $\boldsymbol{u}^{n+1} := [u_1^{n+1}, \ldots, u_M^{n+1}]$.

Similarly to ODEs or Poisson equation, we say that the method is *convergent* if, for a fixed μ, and for every $T > 0$, we have

$$\lim_{\substack{h\to 0, k\to 0\\ k/h^2 = \mu}} \max_{\substack{1\le m\le M\\ 1\le n\le T/k}} |u_m^n - \widehat{u}(mh, nk)| = 0.$$

Theorem 2.1 *If $\mu \le \frac{1}{2}$, then method (2.2) converges.*

Proof. Let $e_m^n := \widehat{u}(mh, nk) - u_m^n$ be the error of approximation, and let $\boldsymbol{e}^n = [e_1^n, \ldots, e_M^n]$ with $\|\boldsymbol{e}^n\|_\infty := \max_m |e_m^n|$. Convergence is equivalent to

$$\lim_{h\to 0} \max_{1\le n\le T/k} \|\boldsymbol{e}^n\|_\infty = 0$$

for every constant $T > 0$. Subtracting (2.1) from (2.2), we obtain

$$\begin{aligned}
e_m^{n+1} &= e_m^n + \mu(e_{m-1}^n - 2e_m^n + e_{m+1}^n) + \eta_m^n\\
&= \mu e_{m-1}^n + (1-2\mu)e_m^n + \mu e_{m+1}^n + \eta_m^n
\end{aligned}$$

where $|\eta_m^n| \le ck^2$ for some constant $c > 0$ (namely $c = c_1 + c_2/\mu$, where $c_1, c_2 > 0$ are defined after equation (2.1)). Then

$$\|\boldsymbol{e}^{n+1}\|_\infty = \max_m |e_m^{n+1}| \le (2\mu + |1-2\mu|)\,\|\boldsymbol{e}^n\|_\infty + ck^2 = \|\boldsymbol{e}^n\|_\infty + ck^2,$$

by virtue of $\mu \le \frac{1}{2}$. Since $\|\boldsymbol{e}^0\|_\infty = 0$, induction yields

$$\|\boldsymbol{e}^n\|_\infty \le cnk^2 \le \tfrac{cT}{k}\,k^2 = cTk \to 0 \qquad (k \to 0).$$

□

Matlab demo: Download the Matlab GUI for *Stability of 1D PDEs* from `http://www.damtp.cam.ac.uk/user/hf323/M21-II-NA/demos/pde_stability/pde_stability.html` and solve the diffusion equation in the interval $[0,1]$ with method (2.2) and $\mu = 0.51 > \frac{1}{2}$. Using (as preset) 100 grid points to discretise $[0,1]$ will then require the time steps to be $5.1 \cdot 10^{-5}$. The solution will evolve very slowly, but wait long enough to see what happens!