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Mathematical Tripos Part II: Michaelmas Term 2023

Numerical Analysis – Lecture 7

Fourier analysis of stability Let us now assume a recurrence of the form

$$\sum_{k=r}^{s} b_k u_{m+k}^{n+1} = \sum_{k=r}^{s} c_k u_{m+k}^{n}, \qquad n \in \mathbb{Z}^+, \quad m \in \mathbb{Z}. \tag{2.5}$$

Within our framework of discretizing PDEs of evolution, this corresponds to $-\infty < x < \infty$ in the underlying PDE and so there are no explicit boundary conditions: this is known as a *Cauchy problem.* The coefficients b_k and c_k are independent of m, n, but typically depend upon μ. We investigate stability by *Fourier analysis.* Note that it doesn't matter what is the underlying PDE: numerical stability is a feature of algebraic recurrences, not of PDEs!

Let $\boldsymbol{v} = (v_m)_{m\in\mathbb{Z}} \in \ell_2[\mathbb{Z}]$. Its *Fourier transform* is the function

$$\widehat{v}(\theta) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}m\theta} v_m, \qquad -\pi \le \theta \le \pi.$$

We equip sequences and functions with the norms

$$\|\boldsymbol{v}\| = \Big\{ \sum_{m\in\mathbb{Z}} |v_m|^2 \Big\}^{\frac{1}{2}} \qquad \text{and} \qquad \|\widehat{v}\|_* = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{v}(\theta)|^2 d\theta \right\}^{\frac{1}{2}}.$$

Lemma 2.11 (Parseval's identity) *For any $\boldsymbol{v} \in \ell_2[\mathbb{Z}]$, we have $\|\boldsymbol{v}\| = \|\widehat{v}\|_*$.*

Proof. By definition,

$$\begin{aligned}
\|\widehat{v}\|_*^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}m\theta} v_m \Big|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} v_m \bar{v}_k \mathrm{e}^{-\mathrm{i}(m-k)\theta} d\theta \\
&= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} v_m \bar{v}_k \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(m-k)\theta} d\theta \overset{(*)}{=} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} v_m \bar{v}_k \delta_{m-k} = \|\boldsymbol{v}\|^2,
\end{aligned}$$

where equality $(*)$ is due to the fact that

$$\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}\ell\theta} d\theta = \begin{cases} 2\pi, & \ell = 0, \\ 0, & \ell \in \mathbb{Z} \setminus \{0\}, \end{cases}$$

□

The implication of the lemma is that the Fourier transform is an *isometry* of the Euclidean norm. This is an important reason underlying its many applications in mathematics and beyond.

Analysis 2.12 (Fourier analysis of stability) For $\theta \in [-\pi, \pi]$, let $\widehat{u}^n(\theta) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}m\theta} u_m^n$ be the Fourier transform of the sequence $\boldsymbol{u}^n \in \ell_2[\mathbb{Z}]$. We multiply the discretized equations (2.5) by $\mathrm{e}^{-\mathrm{i}m\theta}$ and sum up for $m \in \mathbb{Z}$. Thus, the left-hand side yields

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\theta} \sum_{k=r}^{s} b_k u_{m+k}^{n+1} &= \sum_{k=r}^{s} b_k \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\theta} u_{m+k}^{n+1} \\
&= \sum_{k=r}^{s} b_k \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(m-k)\theta} u_m^{n+1} = \left(\sum_{k=r}^{s} b_k \mathrm{e}^{\mathrm{i}k\theta} \right) \widehat{u}^{n+1}(\theta).
\end{aligned}$$

Similarly manipulating the right-hand side, we deduce that

$$\widehat{u}^{n+1}(\theta) = H(\theta) \widehat{u}^n(\theta), \quad \text{where} \quad H(\theta) = \frac{\sum_{k=r}^{s} c_k \mathrm{e}^{\mathrm{i}k\theta}}{\sum_{k=r}^{s} b_k \mathrm{e}^{\mathrm{i}k\theta}}. \tag{2.6}$$

The function H is sometimes called the *amplification factor* of the recurrence (2.5)

Theorem 2.13 *The method (2.5) is stable* $\Leftrightarrow$ $|H(\theta)| \le 1$ *for all* $\theta \in [-\pi, \pi]$.

Proof. The definition of stability is equivalent to the statement that there exists $c > 0$ such that $\|\boldsymbol{u}^n\| \le c$ for all $n \in \mathbb{Z}^+$. The Fourier transform being an isometry, stability is thus equivalent to $\|\widehat{u}^n\|_* \le c$ for all $n \in \mathbb{Z}^+$. Iterating (2.6), we obtain

$$\widehat{u}^n(\theta) = [H(\theta)]^n \widehat{u}^0(\theta), \qquad |\theta| \le \pi, \quad n \in \mathbb{Z}^+. \tag{2.7}$$

1) Assume first that $|H(\theta)| \le 1$ for all $|\theta| \le \pi$. Then, by (2.7),

$$|\widehat{u}^n(\theta)| \le |\widehat{u}^0(\theta)| \qquad \Rightarrow \qquad \|\widehat{u}^n\|_*^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{u}^n(\theta)|^2 \mathrm{d}\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{u}^0(\theta)|^2 \mathrm{d}\theta = \|\widehat{u}^0\|_*^2.$$

Hence stability.

2) Suppose, on the other hand, that there exists $\theta_0 \in [-\pi, \pi]$ such that $|H(\theta_0)| = 1 + 2\epsilon > 1$, say. Since H is continuous, there exist $-\pi \le \theta_1 < \theta_2 \le \pi$ such that $|H(\theta)| \ge 1 + \epsilon$ for all $\theta \in [\theta_1, \theta_2]$. We set $\eta = \theta_2 - \theta_1$ and choose as our initial condition the function (or the $\ell_2[\mathbb{Z}]$-sequence)

$$\widehat{u}^0(\theta) = \begin{cases} \sqrt{\frac{2\pi}{\eta}}, & \theta_1 \le \theta \le \theta_2, \\ 0, & \text{otherwise}, \end{cases}$$

Then

$$\begin{aligned} \|\widehat{u}^n\|_*^2 &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} |H(\theta)|^{2n} |\widehat{u}^0(\theta)|^2 \mathrm{d}\theta = \tfrac{1}{2\pi}\int_{\theta_1}^{\theta_2} |H(\theta)|^{2n} |\widehat{u}^0(\theta)|^2 \mathrm{d}\theta \\ &\ge \tfrac{1}{2\pi}(1+\epsilon)^{2n} \int_{\theta_1}^{\theta_2} \frac{2\pi}{\eta} \mathrm{d}\theta = (1+\epsilon)^{2n} \to \infty \quad (n \to \infty). \end{aligned}$$

We deduce that the method is unstable. $\square$

Examples Consider the Cauchy problem for the diffusion equation.

1) For the Euler method

$$u_m^{n+1} = u_m^n + \mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n)\,,$$

we obtain

$$H(\theta) = 1 + \mu\left(\mathrm{e}^{-\mathrm{i}\theta} - 2 + \mathrm{e}^{\mathrm{i}\theta}\right) = 1 - 4\mu \sin^2 \tfrac{\theta}{2} \;\in\; [1 - 4\mu, 1]\,,$$

thus the method is stable iff $\mu \le \frac{1}{2}$.

2) For the backward Euler method

$$u_m^{n+1} - \mu(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n\,,$$

we have

$$H(\theta) = \left[1 - \mu\left(\mathrm{e}^{-\mathrm{i}\theta} - 2 + \mathrm{e}^{\mathrm{i}\theta}\right)\right]^{-1} = \left[1 + 4\mu \sin^2 \tfrac{\theta}{2}\right]^{-1} \in (0, 1]\,.$$

thus stability for all μ.

3) The Crank–Nicolson scheme

$$u_m^{n+1} - \tfrac{1}{2}\mu(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \tfrac{1}{2}\mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n),$$

results in

$$H(\theta) = \frac{1 + \frac{1}{2}\mu(\mathrm{e}^{-\mathrm{i}\theta} - 2 + \mathrm{e}^{\mathrm{i}\theta})}{1 - \frac{1}{2}\mu(\mathrm{e}^{-\mathrm{i}\theta} - 2 + \mathrm{e}^{\mathrm{i}\theta})} = \frac{1 - 2\mu \sin^2 \frac{\theta}{2}}{1 + 2\mu \sin^2 \frac{\theta}{2}} \in (-1, 1]$$

Hence stability for all $\mu > 0$.