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Mathematical Tripos Part II: Michaelmas Term 2023

Numerical Analysis – Lecture 13

Chebyshev polynomials The Chebyshev polynomial of degree n is defined as

$$T_n(x) := \cos(n \arccos x), \quad x \in [-1, 1], \tag{3.14}$$

or equivalently, by the identity $T_n(\cos\theta) = \cos(n\theta)$ for $\theta \in [0, 2\pi]$.

1) The sequence (T_n) obeys the three-term recurrence relation

$$\begin{aligned} &T_0(x) \equiv 1, \quad T_1(x) = x, \\ &T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \end{aligned}$$

in particular, T_n is indeed an algebraic polynomial of degree n, with the leading coefficient 2^{n-1}. (The recurrence is due to the equality $\cos(n{+}1)\theta + \cos(n{-}1)\theta = 2\cos\theta\cos n\theta$ via substitution $x = \cos\theta$, expressions for T_0 and T_1 are straightforward.)

2) Also, (T_n) forms a sequence of orthogonal polynomials with respect to the inner product $(f, g)_w := \int_{-1}^{1} f(x)g(x)w(x)dx$, with the weight function $w(x) := (1 - x^2)^{-1/2}$. Namely, we have

$$(T_n, T_m)_w = \int_{-1}^{1} T_m(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi} \cos m\theta \cos n\theta\, d\theta = \begin{cases} \pi, & m = n = 0\,, \\ \frac{\pi}{2}, & m = n \geq 1\,, \\ 0, & m \neq n\,. \end{cases} \tag{3.15}$$

Chebyshev expansion Since $(T_n)_{n=0}^{\infty}$ forms an orthogonal sequence, a function f such that $\int_{-1}^{1} |f(x)|^2 w(x)\,dx < \infty$ can be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} \breve{f}_n T_n(x),$$

with the Chebyshev coefficients $\breve{f}_n$. Making inner product of both sides with T_n and using orthogonality yields

$$(f, T_n)_w = \breve{f}_n (T_n, T_n)_w \quad \Rightarrow \quad \breve{f}_n = \frac{(f, T_n)_w}{(T_n, T_n)_w} = \frac{c_n}{\pi}\int_{-1}^{1} f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}}\,, \tag{3.16}$$

where $c_0 = 1$ and $c_n = 2$ for $n \geq 1$.

Connection to the Fourier expansion. Letting $x = \cos t\pi$ and $g(t) = f(\cos(t\pi))$, we obtain

$$\int_{-1}^{1} f(x)T_n(x)\,\frac{dx}{\sqrt{1-x^2}} = \pi\int_0^1 f(\cos t\pi)T_n(\cos t\pi)\,dt = \frac{\pi}{2}\int_{-1}^{1} g(t)\cos nt\pi\,dt\,. \tag{3.17}$$

Given that $\cos nt\pi = \frac{1}{2}(e^{int\pi} + e^{-int\pi})$, and using the Fourier expansion of the 2-periodic function g,

$$g(t) = \sum_{n\in\mathbb{Z}} \widehat{g}_n e^{in\pi t}, \quad \text{where} \quad \widehat{g}_n = \frac{1}{2}\int_{-1}^{1} g(t)e^{-int\pi}\,dt, \quad n \in \mathbb{Z}\,,$$

we continue (3.17) as

$$\int_{-1}^{1} f(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}(\widehat{g}_{-n} + \widehat{g}_n)\,,$$

and from (3.16) we deduce that

$$\breve{f}_n = \begin{cases} \widehat{g}_0, & n = 0\,, \\ \widehat{g}_{-n} + \widehat{g}_n = 2\widehat{g}_n, & n \geq 1\,. \end{cases} \tag{3.18}$$

Convergence speed of Chebyshev expansion Using the connection with Fourier series, one can show that the Chebyshev expansion inherits the exponential convergence provided f can be analytically extended from $[-1,1]$ to the so-called *Bernstein ellipse*.

Theorem 3.17 *Let f be a function on $[-1,1]$ such that it can be extended analytically to the* Bernstein ellipse *in the complex plane*

$$B(a)=\left\{z=x+iy\in\mathbb{C}:\frac{x^2}{\cosh^2(a\pi)}+\frac{y^2}{\sinh^2(a\pi)}<1\right\} \tag{3.19}$$

where $a>0$, and assume furthermore that $|f(z)|\le M$ for $z\in B(a)$. Then with $c=e^{-a\pi}\in(0,1)$, we have $|\breve{f}_n|\le 2Mc^n$ for $n\ge 1$, and $|f(x)-\sum_{n=0}^{N-1}\breve{f}_nT_n(x)|\le 2Mc^N/(1-c)$.

Proof. Let $g(t)=f(\cos(t\pi))=f((e^{it\pi}+e^{-it\pi})/2)$ which is 2-periodic. Let $S(a)=\{z\in\mathbb{C}:|\mathrm{Im}\,z|<a\}$, and note that

$$t\in S(a)\iff\cos(t\pi)\in B(a). \tag{3.20}$$

(See below for justification.) Since f is assumed analytic on $B(a)$, it follows that g is analytic on $S(a)$. From the theorem of Lecture 11, we know that $|\hat{g}_n|\le Me^{-a\pi|n|}$, and thus by (3.18), it follows that $|\breve{f}_0|\le M$ and $|\breve{f}_n|\le 2Me^{-a\pi n}$ for $n\ge 1$. Furthermore, we have, for any $x\in[-1,1]$

$$|f(x)-\sum_{n=0}^{N-1}\breve{f}_nT_n(x)|\le\sum_{n=N}^{\infty}|\breve{f}_n||T_n(x)|\le\sum_{n=N}^{\infty}|\breve{f}_n|\le 2Mc^N/(1-c)$$

as desired.

It remains to prove (3.20). For $b>0$ and $x\in\mathbb{R}$, we have

$$\cos(x+ib)=\tfrac{1}{2}(e^{i(x+ib)}+e^{-i(x+ib)})=\tfrac{1}{2}(e^{-b}e^{ix}+e^{b}e^{-ix})$$

and thus $\mathrm{Re}\,(\cos(x+ib))=\cosh(b)\cos(x)$ and $\mathrm{Im}\,(\cos(x+ib))=-\sinh(b)\sin(x)$. This shows that $\{\cos(x+ib):x\in\mathbb{R}\}$ is precisely the ellipse of equation $x^2/\cosh(b)^2+y^2/\sinh(b)^2=1$. $\square$

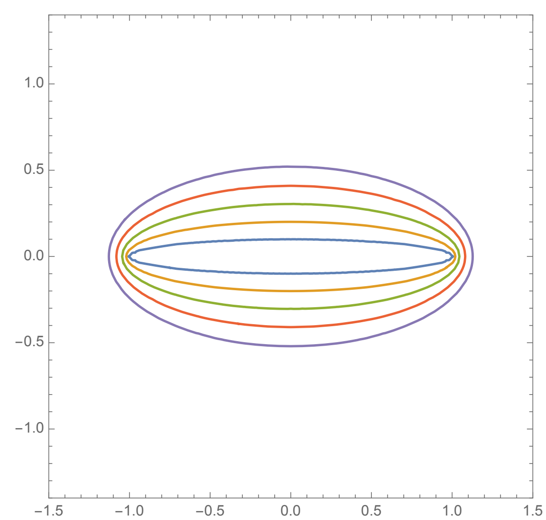

Figure 1: Bernstein ellipses $B(a)$ as defined in (3.19) or different values of $a>0$.

Remark 3.18 (Computing Chebyshev coefficients) *As we have seen, for a general integrable function f, the computation of its Chebyshev expansion is equivalent to the Fourier expansion of the function $g(t)=f(\cos t\pi)$. Since the latter is 2-periodic, we can use a discrete Fourier transform (DFT) to compute the Chebyshev coefficients $\breve{f}_n$. [Actually, based on this connection, one can perform a direct fast Chebyshev transform].*

The algebra of Chebyshev expansions In order to use spectral Galerkin methods with the Chebyshev basis, we need to understand how Chebyshev expansion behaves under pointwise multiplication of functions, and differentiation. Starting by the multiplication operation, we see that

$$\begin{aligned}
T_m(x)T_n(x) &= \cos(m\theta)\cos(n\theta)\\
&= \tfrac{1}{2}\left[\cos((m-n)\theta)+\cos((m+n)\theta)\right]\\
&= \tfrac{1}{2}\left[T_{|m-n|}(x)+T_{m+n}(x)\right]
\end{aligned}$$

and hence,

$$\begin{aligned}
f(x)g(x) &= \sum_{m=0}^{\infty}\breve{f}_m T_m(x)\cdot\sum_{n=0}^{\infty}\breve{g}_n T_n(x) = \frac{1}{2}\sum_{m,n=0}^{\infty}\breve{f}_m\breve{g}_n\left[T_{|m-n|}(x)+T_{m+n}(x)\right]\\
&= \frac{1}{2}\sum_{k=0}^{\infty}T_k(x)\left(\sum_{\substack{m,n\geq 0\\ m+n=k}}\breve{f}_m\breve{g}_n+\sum_{\substack{m,n\geq 0\\ |m-n|=k}}\breve{f}_m\breve{g}_n\right).
\end{aligned}$$

Lemma 3.19 (Derivatives of Chebyshev polynomials) *We can express derivatives T_n' in terms of (T_k) as follows,*

$$T_{2n}'(x) = (2n)\cdot 2\sum_{k=1}^{n}T_{2k-1}(x), \tag{3.21}$$

$$T_{2n+1}'(x) = (2n+1)\left[T_0(x)+2\sum_{k=1}^{n}T_{2k}(x)\right]. \tag{3.22}$$

Proof. From (3.14), we deduce

$$T_m(x)=\cos m\theta \quad\Rightarrow\quad T_m'(x)=\frac{m\sin m\theta}{\sin\theta} \qquad x=\cos\theta\,.$$

So, for $m=2n$, (3.21) follows from the identity $\frac{\sin 2n\theta}{\sin\theta}=2\sum_{k=1}^{n}\cos(2k-1)\theta$, which is verified as

$$2\sin\theta\sum_{k=1}^{n}\cos(2k-1)\theta=\sum_{k=1}^{n}2\cos(2k-1)\theta\sin\theta=\sum_{k=1}^{n}\left[\sin 2k\theta-\sin 2(k-1)\theta\right]=\sin 2n\theta.$$

For $m=2n+1$, (3.22) turns into identity $\frac{\sin(2n+1)\theta}{\sin\theta}=1+2\sum_{k=1}^{n}\cos 2k\theta$, and that follows from

$$\sin\theta\cdot\Big(1+2\sum_{k=1}^{n}\cos 2k\theta\Big)=\sin\theta+\sum_{k=1}^{n}\left[\sin(2k+1)\theta-\sin(2k-1)\theta\right]=\sin(2n+1)\theta.$$

□

The lemma above allows us to express the Chebyshev coefficients of the derivative of a function f, in terms of those of f. We get

$$\begin{cases}
\breve{f'}_0 &= \breve{f}_1+3\breve{f}_3+5\breve{f}_5+\cdots\\
\breve{f'}_1 &= 2(2\breve{f}_2+4\breve{f}_4+6\breve{f}_6+\cdots)\\
\breve{f'}_2 &= 2(3\breve{f}_3+5\breve{f}_5+\cdots)\\
\breve{f'}_3 &= 2(4\breve{f}_4+6\breve{f}_6+\cdots)\\
\vdots
\end{cases}$$

In general, for the k'th derivative we get:

$$\widebreve{f^{(k)}}_n = c_n\sum_{\substack{m=n+1\\ n+m\text{ odd}}}^{\infty} m\;\widebreve{f^{(k-1)}}_m,\quad \forall\, k\geq 1,$$

where $c_0=1$ and $c_n=2$ for $n\geq 1$.