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Mathematical Tripos Part II: Michaelmas Term 2023

Numerical Analysis – Lecture 14

The spectral method for evolutionary PDEs We consider the problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \mathcal{L}u(x,t), & x \in [-1,1], \quad t \geq 0\,, \\ u(x,0) = g(x), & x \in [-1,1], \end{cases} \tag{3.20}$$

We want to solve this problem by the method of lines (semi-discretization), using a spectral method for the approximation of u and its derivatives in the spatial variable x. Then, in a general spectral method, we seek solutions $u_N(x,t)$ with

$$u_N(x,t) = \sum_{n=0}^{N-1} c_n(t)\, \psi_n(x), \tag{3.21}$$

where $c_n(t)$ are expansion coefficients and ψ_n are basis functions.

The spectral approximation in space (3.21) results into a $N \times N$ system of ODEs for the expansion coefficients $\{c_n(t)\}$:

$$\boldsymbol{c}' = B\boldsymbol{c}\,, \tag{3.22}$$

where $B \in \mathbb{R}^{N\times N}$, and $\boldsymbol{c} = \{c_n(t)\} \in \mathbb{R}^N$. We can solve it with standard ODE solvers (Euler, Crank-Nicholson, etc.) which as we have seen are approximations to the matrix exponential in the exact solution $\boldsymbol{c}(t) = \mathrm{e}^{tB}\boldsymbol{c}(0)$.

Example 3.23 (The diffusion equation) Consider the diffusion equation for a function $u = u(x,t)$,

$$\begin{cases} u_t = u_{xx}, & (x,t) \in [-1,1] \times \mathbb{R}_+\,, \\ u(x,0) = g(x), & x \in [-1,1]\,. \end{cases} \tag{3.23}$$

with the periodic boundary conditions $u(-1,t) = u(1,t)$, $u_x(-1,t) = u_x(1,t)$, imposed for all values $t \geq 0$.

For each t, we approximate $u(x,t)$ by its N-th order partial Fourier sum in x,

$$u(x,t) \approx u_N(x,t) \; = \sum_{|n|\leq N/2} \widehat{u}_n(t)\, e^{i\pi n x}.$$

Then, from (3.23), we see that each coefficient $\widehat{u}_n$ fulfills the ODE

$$\widehat{u}_n'(t) = -\pi^2 n^2 \widehat{u}_n(t), \qquad |n| \leq N/2. \tag{3.24}$$

Its exact solution is $\widehat{u}_n(t) = \mathrm{e}^{-\pi^2 n^2 t}\, \widehat{g}_n$, so that

$$u_N(x,t) = \sum_{|n|\leq N/2} \widehat{g}_n\, \mathrm{e}^{-\pi^2 n^2 t}\, \mathrm{e}^{i\pi n x}\,,$$

which is the exact solution truncated to $N+1$ terms.

Here, we were able to find the exact solution without solving the ODE numerically due to the special structure of the Laplacian. However, for more general PDEs we will need a numerical method for which stability has to be analyzed.

Stability analysis The system (3.24) has the form

$$\widehat{\boldsymbol{u}}' = B\widehat{\boldsymbol{u}}\,, \qquad B = \operatorname{diag}\,(-\pi^2 n^2 : n = -N/2, \dots, N/2).$$

If we approximate this system with the Euler method:

$$\widehat{\boldsymbol{u}}^{\ell+1} = (I + kB)\widehat{\boldsymbol{u}}^{\ell}, \qquad k := \Delta t,$$

then the stability condition becomes $\|I + kB\| \le 1$. Since B is diagonal, the same is true for $I + kB$, and the diagonal elements are $1 - k\pi^2 n^2$ with $-N/2 \le n \le N/2$. To have stability, we thus need $1 - k\pi^2 (N/2)^2 \ge -1$, i.e., $k \le 8/(\pi^2 N^2)$.

For the trapezoidal rule, the stability condition will be instead $\|(I - (k/2)B)^{-1}(I + (k/2)B)\| \le 1$ which is satisfied for all $k > 0$, since the spectrum of B is negative.

Example 3.24 (The diffusion equation with non-constant coefficient) We want to solve the diffusion equation with a non-constant coefficient $a(x) > 0$ for a function $u = u(x,t)$

$$\begin{cases} u_t = (a(x) u_x)_x, & (x,t) \in [-1,1] \times \mathbb{R}_+\,, \\ u(x,0) = g(x), & x \in [-1,1]\,. \end{cases} \tag{3.25}$$

Approximating u by its partial Fourier sum results in the following system of ODEs for the coefficients $\widehat{u}_n$

$$\widehat{u}_n'(t) = -\pi^2 \sum_{|m| \le N/2} mn\, \widehat{a}_{n-m}\, \widehat{u}_m(t), \qquad |n| \le N/2\,.$$

For the discretization in time we may apply the Euler method, this gives

$$\widehat{u}_n^{\ell+1} = \widehat{u}_n^{\ell} - k\,\pi^2 \sum_{|m| \le N/2} mn\, \widehat{a}_{n-m}\, \widehat{u}_m^{\ell}\,, \qquad k = \Delta t\,,$$

or in the vector form

$$\widehat{\boldsymbol{u}}^{\ell+1} = (I + kB)\widehat{\boldsymbol{u}}^{\ell},$$

where $B = (b_{m,n}) = (-\pi^2 mn \widehat{a}_{n-m})$. For stability of Euler method, we again need $\|I + kB\| \le 1$.