10 Maximum stable set problem and Lovász ϑ function

In this lecture we will look at another application of semidefinite optimisation to combinatorial optimisation, namely to the maximum stable set problem.

Stable set Let $G = (V, E)$ be an undirected graph. A *stable set* (also known as an *independent set*) in G is a subset $S \subseteq V$ such that no two vertices in S are connected by an edge, i.e., $i, j \in S \Rightarrow \{i, j\} \notin E$. The *maximum stable set problem* is the problem of finding the largest stable set in a graph. The stable set problem can be formulated as the following problem:

$$\begin{array}{lll} \underset{x\in\mathbb{R}^n, X\in\mathbf{S}^n}{\text{maximise}} & \sum_{i=1}^n x_i & \\ \text{subject to} & x_i^2 = x_i & \forall i \in V = \{1, \ldots, n\} \\ & x_i x_j = 0 & \forall ij \in E. \end{array} \tag{1}$$

The constraint $x_i^2 = x_i$ is equivalent to saying that $x_i \in \{0, 1\}$ and the stable set S corresponds to the set of i such that $x_i = 1$. Note that the constraint $x_i x_j = 0$ ensures that S is a stable set. The objective function $\sum_{i=1}^n x_i$ counts the cardinality of S. Solving the optimisation problem (1) is computationally hard in general.

Semidefinite relaxation We are now going to define a semidefinite relaxation for (1). This relaxation was first proposed by Lovász in [Lov79]. It allows us to get an upper bound on the solution (1) by solving a semidefinite program.

$$\begin{array}{lll} \underset{x\in\mathbb{R}^n, X\in\mathbf{S}^n}{\text{maximise}} & \sum_{i=1}^n x_i & \\ \text{subject to} & X_{ii} = x_i & i \in V \\ & X_{ij} = 0 & ij \in E \\ & \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0 & \end{array} \tag{2}$$

Problem (2) can be solved efficiently using algorithms for semidefinite programming. The next theorem shows that (2) yields an upper bound on (1).

Theorem 10.1. *Let $\alpha(G)$ be the solution of (1) and $\vartheta(G)$ be the solution of (2). Then $\alpha(G) \leq \vartheta(G)$.*

Proof. It suffices to observe that if x is feasible for (1), then the pair $(x, X = xx^T)$ is feasible for (2) since

$$\begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \succeq 0.$$

□

A natural question is to ask whether there is a constant $c > 0$ such that $c \cdot \vartheta(G) \leq \alpha(G)$ for all graphs G, like for the maximum cut problem that we saw last time. Unfortunately this is not the case. Indeed one can show:

Theorem 10.2. *There exists a sequence of graphs (G_n) such that $\alpha(G_n)/\vartheta(G_n) \to 0$ as $n \to \infty$.*

Proof. See Exercise 10.1. □

Exercise 10.1 (Proof of Theorem 10.2)**.** *In this exercise we will prove Theorem 10.2. In fact we will show something more precise. We will prove that for any large enough n there is a graph G on n nodes such that $\alpha(G)/\vartheta(G) \le O(\log(n)/\sqrt{n})$ as $n \to \infty$.*

1. *Show that the dual of (2) can be expressed as*

$$\begin{array}{ll} \text{min.} & Z_{00} \\ \text{s.t.} & z_i = (1 + Z_{ii})/2 \qquad \forall i \in V \\ & Z_{ij} = 0 \qquad \forall \{i,j\} \in \overline{E} \\ & \begin{bmatrix} Z_{00} & z^T \\ z & Z \end{bmatrix} \succeq 0 \end{array} \tag{3}$$

   where $\overline{E} = \{\{i,j\} : i \neq j \text{ and } \{i,j\} \notin E\}$ is the complement of E [hint: you may need to use the fact that $\left[\begin{smallmatrix} A & B^T \\ B & C \end{smallmatrix}\right] \succeq 0 \iff \left[\begin{smallmatrix} A & -B^T \\ -B & C \end{smallmatrix}\right] \succeq 0$].

2. *Show that (3) can be simplified to:*

$$\begin{array}{ll} \text{min.} & Z_{00} \\ \text{s.t.} & Z_{ii} = 1 \qquad \forall i \in V \\ & Z_{ij} = 0 \qquad ij \in \overline{E} \\ & \begin{bmatrix} Z_{00} & \mathbf{1}^T \\ \mathbf{1} & Z \end{bmatrix} \succeq 0 \end{array} \tag{4}$$

   where $\mathbf{1}$ denotes the vector with all ones [hint: given (z, Z) feasible for (3), consider $\tilde{Z}_{ij} = Z_{ij}/(z_i z_j)$].

3. *Use Slater condition to verify that (4) and (2) have the same optimal values.*

4. *Show that for any graph G with n vertices we have $\vartheta(G)\vartheta(\bar{G}) \ge n$ where $\bar{G} = (V, \bar{E})$ [hint: use the minimisation formulation (4) of $\vartheta(G)$ to construct a feasible point for (2) applied to $\bar{G}$]. Deduce that for any graph G we have either $\vartheta(G) \ge \sqrt{n}$ or $\vartheta(\bar{G}) \ge \sqrt{n}$.*

5. *We are now going to assume the following fact: for any n large enough (i.e., $n \ge N_0$ for some N_0) there is a graph G on n vertices such that $\max(\alpha(G), \alpha(\bar{G})) \le 3\log(n)$. Using this fact together with question 4, prove the desired result.*

Note: One way to prove the existence of a graph such that $\max(\alpha(G), \alpha(\bar{G})) \le 3\log(n)$ is using the probabilistic method. If we let G be a random undirected graph on $V = \{1, \dots, n\}$ where we draw an edge between a pair $\{i,j\} \subset V$ with probability 1/2 (independently of the other pairs) a well-known result states that $\alpha(G)$ concentrates around $2\log(n)$.

References

[Lov79] László Lovász. On the Shannon capacity of a graph. *IEEE Transactions on Information Theory*, 25(1):1–7, 1979. 1