11 Nonnegative polynomials, sums of squares and semidefinite programming

Definition 11.1. We say a polynomial $p \in \mathbb{R}[x]$ is *nonnegative* if $p(x) \geq 0$ for all $x \in \mathbb{R}$.

Note that if we have a "good" way of checking nonnegativity of polynomials then we can also minimise (or maximise) polynomials. Indeed if $p(x)$ is a polynomial then

$$\min_{x\in\mathbb{R}} p(x) \quad = \quad \max \quad \gamma \quad \text{s.t.} \quad p - \gamma \text{ is nonnegative.} \tag{1}$$

It is not difficult to verify that if p is nonnegative then:

- The degree of p is even and the leading coefficient (i.e., the coefficient of x^{2d} if $\deg p = 2d$) is nonnegative.
- Any real root of p has even multiplicity.

These conditions can also be shown to be sufficient (see proof of Theorem 11.1 below).

Definition 11.2. We say that a polynomial $p \in \mathbb{R}[x]$ is a sum-of-squares if there exist polynomials $q_1, \ldots, q_k \in \mathbb{R}[x]$ such that $p = \sum_{i=1}^k q_i^2$.

It is clear that if p is a sum of squares, then it is globally nonnegative. The converse is also true for polynomials in one variable.

Theorem 11.1. *A univariate polynomial $p(x) = \sum_{k=0}^{2d} a_k x^k$ of degree $2d$ is globally nonnegative if and only if there exist q_1, q_2 of degree d such that $p = q_1^2 + q_2^2$.*

Proof. The implication $\Leftarrow$ is clear. Assume $p(x)$ is nonnegative. Since p has real coefficients, if $p(z) = 0$ then $p(\bar{z}) = 0$. Furthermore if z is a real root of p then it must have even multiplicity. This implies that we can write:

$$p(x) = a_{2d} \prod_{i=1}^{d} (x - z_i)(x - \bar{z}_i) = |q(x)|^2$$

where $q(x) = \sqrt{a_{2d}} \prod_{i=1}^d (x - z_i)$ (note that $a_{2d} \geq 0$ since p is nonnegative). If we let $q_1(x) = \mathrm{Re}[q(x)]$ and $q_2 = \mathrm{Im}[q(x)]$ (one can easily verify that these are polynomials of degree at most d) we get the desired result. □

The next theorem shows that checking nonnegativity of a polynomial in one variable can be done using semidefinite programming:

Theorem 11.2. *A polynomial $p(x) = \sum_{k=0}^{2d} a_k x^k$ is nonnegative if, and only if, there exists a positive semidefinite matrix M of size $(d+1) \times (d+1)$ such that*

$$a_k = \sum_{\substack{0 \leq i,j \leq d \\ i+j=k}} M_{ij} \qquad \forall k = 0, \ldots, 2d. \tag{2}$$

(The rows and columns of the matrix M are indexed by $0, \ldots, d$ instead of $1, \ldots, d+1$ for convenience.)

Proof. We first prove $\Leftarrow$. Assume there exists a matrix $M \in \mathbf{S}_+^{d+1}$ that satisfies (2). Then by definition of M being positive semidefinite we have, letting $[x] = (1, x, \ldots, x^d)$:

$$0 \leq [x]^T M[x] = \sum_{0 \leq i,j \leq d} M_{ij} x^i x^j = \sum_{0 \leq i,j \leq d} M_{ij} x^{i+j} = \sum_{k=0}^{2d} \left(\sum_{\substack{0 \leq i,j \leq d \\ i+j=k}} M_{ij} \right) x^k = p(x).$$

To prove the converse, assume p is nonnegative. By Theorem 11.2 we know there exist polynomials $q_1, q_2 \in \mathbb{R}[x]$ of degree d such that $p(x) = q_1(x)^2 + q_2(x)^2$. Write $(c_0, \ldots, c_d) \in \mathbb{R}^{d+1}$ (resp. $(e_0, \ldots, e_d) \in \mathbb{R}^{d+1}$) the coefficients of q_1 (resp. q_2) in the monomial basis. Using the notation $[x] = (1, x, \ldots, x^d)$ we have that $q_1(x)^2 = (c^T[x])^2$ and $q_2(x)^2 = (e^T[x])^2$. Thus $p(x) = [x]^T(cc^T + ee^T)[x] = [x]^T M[x]$ where we defined $M = cc^T + ee^T \succeq 0$. Equating the coefficients of $p(x)$ and $[x]^T M[x]$ in the monomial basis we get (2) as desired. $\square$

The previous theorem says that checking if a polynomial is nonnegative is a semidefinite feasibility problem. In turn it also allows us to express the minimisation problem (1) as a semidefinite program (see Exercise 11.1).

Example 1 (Polynomials of degree 2)**.** *We know from high-school algebra that a polynomial $p(x) = ax^2 + bx + c$ is nonnegative iff $b^2 - 4ac \leq 0$ and $a, c \geq 0$. Theorem 11.2 tells us that this polynomial is nonnegative if and only if there exists a matrix $M \in \mathbf{S}^2$ such that*

$$\begin{aligned} &M \succeq 0, \\ &M_{00} = c, \\ &M_{01} + M_{10} = b, \\ &M_{11} = a. \end{aligned}$$

This is equivalent to saying that $\begin{bmatrix} c & b/2 \\ b/2 & a \end{bmatrix}$ is positive semidefinite, which in turn is equivalent to having $b^2 - 4ac \leq 0$ and $a, c \geq 0$.

Exercise 11.1. *Write a semidefinite program that computes the minimum, over $\mathbb{R}$, of the polynomial $p(x) = x^4 + 3x^3 - x^2 + x - 1$. Implement and solve your semidefinite program using CVX.*

Exercise 11.2 (Nonnegativity on intervals)**.** *The purpose of this exercise is to prove variants of Theorem 11.1 for polynomials p nonnegative on an interval.*

1. *Show that a polynomial $p \in \mathbb{R}[x]$ satisfies $p(x) \geq 0$ for all $x \in [0, \infty)$ if and only if there exist $s_1, s_2 \in \mathbb{R}[x]$ sums-of-squares such that*

   $$p(x) = s_1(x) + x s_2(x)$$

   with the following degree bounds: $\deg s_1 \leq 2d$ and $\deg s_2 \leq 2d - 2$ if $\deg p = 2d$ (even); and $\deg(s_1) \leq 2d$ and $\deg(s_2) \leq 2d$ if $\deg(p) = 2d + 1$ (odd).

2. *Let $a \leq b$. Show that a polynomial $p \in \mathbb{R}[x]$ with even degree $\deg p = 2d$ satisfies $p(x) \geq 0$ on $[a, b]$ if and only if there exist $s_1, s_2 \in \mathbb{R}[x]$ sums-of-squares with $\deg s_1 \leq 2d$ and*

$\deg s_2 \leq 2d - 2$ *such that*

$$p(x) = s_1(x) + (b - x)(x - a)s_2(x).$$

When $\deg p = 2d + 1$ *(odd) show that* $p(x) \geq 0$ *on* $[a, b]$ *if and only if there exist polynomials* $s_1, s_2 \in \mathbb{R}[x]$ *sums-of-squares with* $\deg s_1 \leq 2d$ *and* $\deg s_2 \leq 2d$ *such that*

$$p(x) = (x - a)s_1(x) + (b - x)s_2(x).$$