12 The cone of nonnegative univariate polynomials

Let P_{2d} be the cone of nonnegative polynomials (in one variable) of degree $2d$:

$$P_{2d} = \left\{ (p_0, \ldots, p_{2d}) \ : \ \sum_{k=0}^{2d} p_k x^k \geq 0 \ \ \forall x \in \mathbb{R} \right\}.$$

Theorem 12.1. *P_{2d} is a proper cone.*

Proof. We have to check that P_{2d} is closed, convex, pointed and has nonempty interior. Checking that P_{2d} is convex, closed and pointed is easy. We leave it as an exercise to verify that the polynomial $x^{2d} + 1$ is in the interior of P_{2d}. □

We saw last time that P_{2d} has the following *semidefinite representation*:

$$p \in P_{2d} \iff \exists M \in \mathbf{S}_+^{d+1} \text{ s.t. } \sum_{\substack{0 \leq i,j \leq d \\ i+j=k}} M_{ij} = p_k. \qquad (1)$$

This means that any conic program over P_{2d} is actually a semidefinite program.

Example We now look at a simple example of polynomial optimisation. Let p be a polynomial and consider the problem of computing the minimum of p on the interval $[-1, 1]$. We know that

$$\min_{x \in [-1,1]} p(x) \quad = \quad \max \quad \gamma \quad \text{s.t.} \quad p - \gamma \text{ nonnegative on } [-1, 1]. \qquad (2)$$

The following result (which appears as Exercise 11.2 in Lecture 11) gives necessary and sufficient conditions for a polynomial to be nonnegative on $[-1, 1]$.

Theorem 12.2 (Nonnegative polynomials on $[-1, 1]$)**.** *A polynomial p of even degree $2d$ is nonnegative on $[-1, 1]$ if and only if there exist $s_1 \in P_{2d}$ and $s_2 \in P_{2d-2}$ such that $p(x) = s_1(x) + (1 - x^2)s_2(x)$.*

A polynomial p of odd degree $2d + 1$ is nonnegative on $[-1, 1]$ if and only if there exist $s_1 \in P_{2d}$ and $s_2 \in P_{2d}$ such that $p(x) = (1 - x)s_1(x) + (1 + x)s_2(x)$.

Using this theorem we can rewrite the problem (2) as follows (we assume for this example that $\deg p = 2d$ is even):

$$\begin{array}{ll} \underset{\gamma \in \mathbb{R}, s_1 \in \mathbb{R}^{2d+1}, s_2 \in \mathbb{R}^{2d-1}}{\text{maximise}} & \gamma \\ \text{subject to} & p(x) - \gamma = s_1(x) + (1 - x^2)s_2(x) \\ & s_1 \in P_{2d} \\ & s_2 \in P_{2d-2} \end{array} \qquad (3)$$

The first constraint in (3) says that the polynomials $p(x) - \gamma$ and $s_1(x) + (1 - x^2)s_2(x)$ must be equal, i.e., have the same coefficients. Writing out this constraint explicitly we see that it consists of *linear equalities* in γ and the coefficients of s_1 and s_2 (it is important to understand that the "x" that appears in the first constraint of (3) is *not* a variable of the optimisation problem; it is just an indeterminate). Since P_{2d} and P_{2d-2} admit a semidefinite representation, we see that (3) is a semidefinite program. The following code implements the problem (3) on CVX (we use CVX's built-in function `nonneg_poly_coeffs(2*d)` which internally represents the cone P_{2d} using (1)).

```
% Find the minimum of p(x) on [-1,1]
% p(x) = 4x^4 + 3x^3 - 2*x^2 + 2
p = [4 3 -2 0 2]';
d = (length(p)-1)/2;
cvx_begin
    variable g % gamma
    variable s1(2*d+1)      % polynomial of degree 2d
    variable s2(2*d-1)      % polynomial of degree 2d-2
    maximize g
    subject to
        % p(x) - gamma = s_1(x) + (1-x^2)*s_2(x)
        p - [zeros(2*d,1); g] == s1 + conv( [-1 ; 0 ; 1] , s2 );
        s1 == nonneg_poly_coeffs(2*d);    % s_1 \in P_{2d}
        s2 == nonneg_poly_coeffs(2*d-2);  % s_2 \in P_{2d-2}
cvx_end
```

Duality The dual cone of P_{2d} is, by definition:

$$P_{2d}^* = \left\{ (y_0, \dots, y_{2d}) \in \mathbb{R}^{2d+1} \quad : \quad \sum_{k=0}^{2d} p_k y_k \geq 0 \quad \forall p \in P_{2d} \right\}.$$

It is not difficult to produce certain vectors in P_{2d}^*. For example if $x_0 \in \mathbb{R}$ then the vector

$$y_{x_0} = (1, x_0, x_0^2, \dots, x_0^{2d}) \tag{4}$$

lives in P_{2d}^*. This is because if $p \in P_{2d}$ then the inner product $\langle p, y_{x_0} \rangle$ is nothing but $p(x_0)$ which is nonnegative since p is globally nonnegative. It turns out that, up to closure, any element of P_{2d}^* is a nonnegative combination of vectors of the form (4). Let M_{2d} be the convex cone generated by the vectors $\{y_{x_0}\}_{x_0 \in \mathbb{R}}$:

$$M_{2d} = \operatorname{cone}(y_{x_0} : x_0 \in \mathbb{R}).$$

Theorem 12.3. $P_{2d}^* = \operatorname{cl} M_{2d}$.

Proof. We already saw the inclusion $\supseteq$. Assume for contradiction the other inclusion is not true. Then there is a point $y \in P_{2d}^*$ that is not in the closed conic hull of the y_{x_0}'s. By the separating hyperplane theorem this means that there exists p such that $\langle p, y \rangle < 0$ and $\langle p, y_x \rangle \geq 0$ for all y_x. Since $\langle p, y_x \rangle = p(x)$, the last condition tells us that the polynomial $p(x) = \sum_{k=0}^{2d} p_k x^k$ is nonnegative on $\mathbb{R}$. That $\langle p, y \rangle < 0$ contradicts the fact that $y \in P_{2d}^*$. □

The cone M_{2d} is not closed in general and that is why we need the closure in the statement of the Theorem 12.3. (Recall that a dual cone is *always* closed (and convex) since it is, by definition, the intersection of closed halfspaces.) For example one can verify $(0,0,1) \in \operatorname{cl}(M_2) \setminus M_2$: indeed, on the one hand it is not possible to write $(0,0,1)$ as a conic combination of the $\{y_x : x \in \mathbb{R}\}$, and on other hand we have $(0,0,1) = \lim_{x \to \infty} \frac{1}{x^2} y_x$.

Exercise 12.1. *Given $p \in \mathbb{R}[x]$ let $\|p\|_\infty = \max_{x \in [-1,1]} |p(x)|$. Given an integer $n \geq 1$ we are interested in finding the minimum of $\|p\|_\infty$ over all* monic *polynomials p of degree n (recall that a polynomial is called monic if its leading coefficient is equal to 1, where the leading coefficient is the coefficient of the monomial x^n if $n = \deg(p)$).*

Show how to formulate this problem using the cone of nonnegative polynomials, and solve *it using CVX. What optimal values do you get for different choices of n? Can you recognise the polynomial that achieves the optimal value?*

Exercise 12.2. *Show how to formulate the cone of* convex *polynomials using the cone of nonnegative polynomials.*