14 Nonnegative multivariate polynomials

We start looking in this lecture at polynomials in more than one variable. We first fix some notations. We denote by $\mathbb{R}[x_1,\dots,x_n]$ the space of polynomials in n variables $x_1,\dots,x_n$. A *monomial* is an expression $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ where $\alpha_1,\dots,\alpha_n$ are integers. We will often use the shorthand notation $\mathbf{x}^\alpha = x_1^{\alpha}\dots x_n^{\alpha_n}$ where $\mathbf{x}=(x_1,\dots,x_n)$ and $\alpha=(\alpha_1,\dots,\alpha_n)$. The degree of a monomial $\mathbf{x}^\alpha$ is $|\alpha| := \alpha_1+\cdots+\alpha_n$. The degree of a polynomial is the largest degree of its monomials. For example the polynomial $p(x_1,x_2) = x_1x_2^2 + x_1x_2 + 1$ has degree 3. We will also use the notation $\mathbb{R}[x_1,\dots,x_n]_{\leq d}$ for the space of polynomials of degree at most d.

We are interested in polynomials $p(x_1,\dots,x_n)$ that are nonnegative on $\mathbb{R}^n$, i.e., such that $p(x_1,\dots,x_n)\geq 0$ for all $x\in\mathbb{R}^n$. An obvious sufficient condition for a polynomial to be nonnegative is for it to be a sum of squares.

Definition 14.1. A polynomial $p\in\mathbb{R}[\mathbf{x}]$ is a *sum of squares* if there exist polynomials $q_1,\dots,q_k\in\mathbb{R}[\mathbf{x}]$ such that $p(\mathbf{x}) = q_1(\mathbf{x})^2+\cdots+q_k(\mathbf{x})^2$.

Exercise 14.1. *Show that if $p\in\mathbb{R}[x_1,\dots,x_n]$ is nonnegative then p has even degree. Show that if $\deg p = 2d$ and $p(\mathbf{x}) = \sum_{i=1}^k q_i(\mathbf{x})^2$ then necessarily $\deg q_i \leq d$ for each $i=1,\dots,k$.*

Exercise 14.2. *Show that the space $\mathbb{R}[x_1,\dots,x_n]_{\leq d}$ of polynomials of degree at most d has dimension $\binom{n+d}{d}$.*

We saw that for polynomials of one variable ($n=1$) any nonnegative polynomial is a sum-of-squares. It turns out that in general this is not the case. Let $P_{n,2d}$ be the cone of nonnegative polynomials in n variables of degree at most $2d$. Let $\Sigma_{n,2d}$ be the cone of polynomials of degree at most $2d$ that are sums of squares.

Theorem 14.1 (Hilbert). *$P_{n,2d}=\Sigma_{n,2d}$ if and only if $n=1$ or $2d=2$ or $(n,2d)=(2,4)$.*

We have already seen that $P_{n,2d}=\Sigma_{n,2d}$ in the case $n=1$. The case $2d=2$ can be proved using, e.g., eigenvalue decomposition of symmetric positive semidefinite matrices. The last case $(n,2d)=(2,4)$ is more difficult. For more on this problem and the cases where $P_{n,2d}\neq\Sigma_{n,2d}$, we refer to [Rez00] and [BPT12, Chapter 4].

Checking whether a general polynomial is nonnegative is hard computationally. On the other hand checking whether a polynomial is a sum-of-squares can be done using semidefinite programming. This is the object of the next theorem and it is the analogue of Theorem 11.2 in the multivariate setting. For convenience we let $s(n,d) = \dim\mathbb{R}[x_1,\dots,x_n]_{\leq d} = \binom{n+d}{d}$.

Theorem 14.2. *Let $p(x)\in\mathbb{R}[x_1,\dots,x_n]$ of degree $2d$ with expansion:*

$$p(\mathbf{x}) = \sum_{\gamma\in\mathbb{N}^n : |\gamma|\leq 2d} p_\gamma \mathbf{x}^\gamma$$

Then $p(\mathbf{x})$ is a sum-of-squares if and only if there exists a matrix $Q \in \mathbf{S}^{s(n,d)}$ such that $Q \succeq 0$ and

$$\sum_{\substack{\alpha,\beta\in\mathbb{N}^n\\|\alpha|,|\beta|\le d\\\alpha+\beta=\gamma}} Q_{\alpha,\beta} = p_\gamma \quad \forall\gamma\in\mathbb{N}^n, |\gamma|\le 2d. \tag{1}$$

Proof. Throughout the proof we use the notation $[\mathbf{x}]_d$ for the vector of size $s(n,d)$ containing all monomials of degree at most d. For example if $n=2$ and $d=2$ then $[\mathbf{x}]_d = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2)$. We prove the theorem in a sequence of equivalences (some of the steps are explained below):

$$\begin{aligned}
p \text{ is sum-of-squares} &\iff \exists q_1,\dots,q_k \in \mathbb{R}[x_1,\dots,x_n]_{\le d} \text{ s.t. } p = \sum_{i=1}^k q_i^2\\
&\overset{(a)}{\iff} \exists q_1,\dots,q_k \in \mathbb{R}^{s(n,d)} \text{ s.t. } p(\mathbf{x}) = \sum_{i=1}^k (\langle q_i, [\mathbf{x}]_d\rangle)^2\\
&\iff \exists q_1,\dots,q_k \in \mathbb{R}^{s(n,d)} \text{ s.t. } p(\mathbf{x}) = \sum_{i=1}^k \langle q_iq_i^T, [\mathbf{x}]_d[\mathbf{x}]_d^T\rangle\\
&\iff \exists q_1,\dots,q_k \in \mathbb{R}^{s(n,d)} \text{ s.t. } p(\mathbf{x}) = \left\langle \sum_{i=1}^k q_iq_i^T, [\mathbf{x}]_d[\mathbf{x}]_d^T\right\rangle\\
&\overset{(b)}{\iff} \exists Q \in \mathbf{S}_+^{s(n,d)} \text{ s.t. } p(\mathbf{x}) = \langle Q, [\mathbf{x}]_d[\mathbf{x}]_d^T\rangle\\
&\overset{(c)}{\iff} \exists Q \in \mathbf{S}_+^{s(n,d)} \text{ s.t. } p_\gamma = \sum_{\substack{\alpha,\beta\in\mathbb{N}^n\\|\alpha|,|\beta|\le d\\\alpha+\beta=\gamma}} Q_{\alpha,\beta}\ \forall\gamma\in\mathbb{N}^n, |\gamma|\le 2d.
\end{aligned}$$

In (a) we identified polynomials $q_1,\dots,q_k$ with their vector of coefficients $q_1,\dots,q_k\in\mathbb{R}^{s(n,d)}$. In (b) we let $Q=\sum_{i=1}^k q_iq_i^T$. The last step (c) is obtained by matching coefficients in $p(\mathbf{x}) = \langle Q, [\mathbf{x}]_d[\mathbf{x}]_d^T\rangle$; indeed we have:

$$[\mathbf{x}]_d^T Q[\mathbf{x}] = \sum_{\alpha,\beta} Q_{\alpha,\beta}\mathbf{x}^\alpha\mathbf{x}^\beta = \sum_\gamma \left(\sum_{\alpha,\beta:\alpha+\beta=\gamma} Q_{\alpha,\beta}\right)\mathbf{x}^\gamma.$$

□

Example 1. *Let us look at a concrete example of polynomial. This example is taken from [BPT12, Example 3.38, page 64]. We want to decide whether the polynomial*

$$p(x,y) = 2x^4 + 5y^4 - x^2y^2 + 2x^3y + 2x + 2$$

is a sum of squares. Here $n=2$ and $2d=4$ so our matrix Q will be indexed by monomials of degree $d=2$ in $n=2$ variables

$$Q = \begin{bmatrix}
q_{00,00} & q_{00,10} & q_{00,01} & q_{00,20} & q_{00,11} & q_{00,02}\\
 & q_{10,10} & q_{10,01} & q_{10,20} & q_{10,11} & q_{10,02}\\
 & & q_{01,01} & q_{01,20} & q_{01,11} & q_{01,02}\\
 & & & q_{20,20} & q_{20,11} & q_{20,02}\\
 & & & & q_{11,11} & q_{11,02}\\
 & & & & & q_{02,02}
\end{bmatrix}. \tag{2}$$

(We only wrote the entries above the diagonal since the matrix Q is symmetric.) Checking whether $p(x,y)$ is a sum-of-squares is equivalent to checking whether there is a matrix Q of the form (2) that satisfies the linear constraints (1). In our case there is a total of $s(n,2d) = s(2,4) = \binom{6}{4} = 15$ linear equations, one for each monomial $\mathbf{x}^\gamma$ of degree at most $2d$. We only write some of these equations below just to give an idea: (the equations below are the ones we get for the monomials $\gamma = (4,0)$, $\gamma = (2,2)$ and $\gamma = (0,2)$)

$$\begin{aligned} x^4 \quad (\gamma = (4,0)): &\quad 2 = q_{20,20} \\ x^2y^2 \quad (\gamma = (2,2)): &\quad -1 = 2q_{20,02} + q_{11,11} \\ y^2 \quad (\gamma = (0,2)): &\quad 0 = 2q_{00,02} + q_{01,01}. \end{aligned}$$

Checking feasibility of the resulting semidefinite program will tell us that $p(x,y)$ is indeed a sum of squares. See [BPT12, Example 3.38, page 64] for an explicit sum-of-squares decomposition of $p(x,y)$.

Application: Global optimisation of polynomials Consider the problem of minimising a given polynomial $p(\mathbf{x})$ over $\mathbb{R}^n$. We saw in previous lectures that

$$\min_{\mathbf{x}\in\mathbb{R}^n} p(\mathbf{x}) \quad = \quad \max_{\gamma\in\mathbb{R}} \gamma \quad \text{s.t.} \quad p - \gamma \text{ is nonnegative.} \tag{3}$$

We can relax the latter problem and replace the constraint "$p-\gamma$ nonnegative", by "$p-\gamma$ is a sum of squares":

$$\max_{\gamma\in\mathbb{R}} \gamma \quad \text{s.t.} \quad p - \gamma \text{ is a sum-of-squares.} \tag{4}$$

The optimisation problem (4) can be formulated as a semidefinite program via Theorem 14.2. The optimal value of that semidefinite program gives us a *lower bound* to our problem (3).

Exercise 14.3. *Compute a lower bound on the minimum of the polynomial $p(x,y) = x^2 - 2xy + 2y^2 + 2x + 4y + 8$ using the sum-of-squares relaxation. Is the lower bound you get tight?*

References

[BPT12] Grigoriy Blekherman, Pablo A. Parrilo, and Rekha R. Thomas. *Semidefinite optimization and convex algebraic geometry*. SIAM, 2012. 1, 2, 3

[Rez00] Bruce Reznick. Some concrete aspects of hilbert's 17th problem. *Contemporary Mathematics*, 253:251–272, 2000. 1