2 Review of convexity (continued)

We saw last time that any closed bounded convex set in $\mathbb{R}^n$ is the convex hull of its extreme points (Minkowski's theorem). In the next theorem we show that any closed convex set can be expressed as an intersection of halfspaces.

Theorem 2.1. *Assume C is a closed convex subset of $\mathbb{R}^n$. Then C is the intersection of all halfspaces that contain it, i.e., we have*

$$C = \bigcap_{\substack{H \text{ halfspace} \\ C \subseteq H}} H. \tag{1}$$

Proof. This is a direct application of the separating hyperplane theorem. Let the right-hand side of (1) be D. It is trivial that $C \subseteq D$. To show that $D \subseteq C$ we proceed by contrapositive, i.e., we will show that if $x \notin C$ then $x \notin D$. If $x \notin C$ by the separating hyperplane theorem there exists $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$ such that $\langle a, z \rangle < b$ for $z \in C$ and $\langle a, x \rangle > b$ (we used here a *strict* version of the separating hyperplane theorem which holds when C is closed). If we call H the halfspace $H = \{z \in \mathbb{R}^n : \langle a, z \rangle \leq b\}$ we have $C \subseteq H$. Thus by definition of D we have $D \subseteq H$. Since $x \notin H$ it follows $x \notin D$ which is what we wanted. $\square$

Summary: We have thus seen that any closed and bounded convex subset of $\mathbb{R}^n$ has two *dual* descriptions: an "internal" description as a convex hull of points (Minkowski's theorem); and an "external" description as an intersection of halfspaces (Theorem 2.1). This is a manifestation of *duality theory* in convex analysis/geometry.

Definition 2.1 (Cone)**.** A set $K \subseteq \mathbb{R}^n$ is called a *cone* if for any $x \in K$ and $\lambda \geq 0$ we have $\lambda x \in K$. The cone is called *pointed* if $K \cap (-K) = \{0\}$.

Note that a set K is a *convex cone* if and only if for any $x, y \in K$, $x + y \in K$. A *conic combination* of a set of vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is a linear combination of the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$ where $\lambda_1, \ldots, \lambda_k$ are nonnegative. Examples of convex cones:

- Nonnegative orthant: $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \ \forall i = 1, \ldots, n\}$.
- "Ice-cream cone": $\mathbf{Q}^{n+1} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\|_2 \leq t\}$.

Definition 2.2 (Dual cone)**.** If K is a cone in $\mathbb{R}^n$ the *dual cone* K^* is defined as:

$$K^* = \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \ \forall x \in K\} \tag{2}$$

Theorem 2.2. *Let K be a cone in $\mathbb{R}^n$. Then K^* is a closed convex cone. Furthermore if K itself is closed and convex then $(K^*)^* = K$.*

Proof. Note that K^* can be expressed as

$$K^* = \bigcap_{x \in K} \underbrace{\{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0\}}_{H_x}.$$

Each H_x is a closed halfspace, thus K^* is closed and convex as an intersection of closed convex sets. The proof that $(K^*)^* = K$ when K is closed and convex is similar to the proof of Theorem 2.1. We leave it as an exercise. $\square$

We now define *extreme rays* of cones, which play the same role as *extreme points* for bounded closed convex sets.

Definition 2.3 (Extreme ray of a cone)**.** An *extreme ray* of a cone $K \subseteq \mathbb{R}^n$ is a subset $S \subseteq K$ of the form $S = \mathbb{R}_+ v = \{\lambda v : \lambda \geq 0\}$ where $v \neq 0$ that satisfies the following: for any $x, y \in K$ if $x + y \in S$ then $x, y \in S$.

We will sometimes abuse notation and say that a vector $v \in K \setminus \{0\}$ is an *extreme ray* of K if $\mathbb{R}_+ v$ is an extreme ray of K.

Definition 2.4 (Conical hull)**.** Let $S \subseteq \mathbb{R}^n$. The *conical hull* of S is the smallest convex cone that contains S, i.e.,

$$S = \bigcap_{\substack{K \text{ convex cone} \\ S \subseteq K}} K = \left\{ x \in \mathbb{R}^n : \exists k \in \mathbb{N}_{\geq 1}, s_1, \ldots, s_k \in S, \lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0} \text{ s.t. } x = \sum_{i=1}^{k} \lambda_i s_i \right\}.$$

Minkowski's theorem for cones can then be stated as:

Theorem 2.3 (Minkowski's theorem for closed convex pointed cones)**.** *Assume K is a closed and pointed convex cone in $\mathbb{R}^n$. Then K is the* conical hull *of its extreme rays, i.e., any element in K can be expressed as a conic combination of its extreme rays.*

Proof. See Exercise 2.2 for a proof. □

Exercise 2.1 (Properties of cones and their duals)**.** *Let K be a closed convex cone in $\mathbb{R}^n$.*

1. *Show that the following are equivalent:*

   (i) K has nonempty interior

   (ii) $\text{span}(K) = \mathbb{R}^n$

   (iii) For any $w \in \mathbb{R}^n \setminus \{0\}$ there exists $x \in K$ such that $\langle w, x \rangle \neq 0$.

2. *Show that K is pointed if and only K^* has nonempty interior.*

3. *Show that $y \in \text{int}(K^*)$ if and only if $\langle y, x \rangle > 0$ for all $x \in K \setminus \{0\}$.*

Exercise 2.2 (Proof of Minkowski's theorem for closed convex pointed cones)**.** *In this exercise we are going to prove Theorem 2.3. Let K be a closed pointed convex cone.*

1. *We are first going to assume that there exists $y \in \mathbb{R}^n$ such that $\langle y, x \rangle > 0$ for all $x \in K \setminus \{0\}$. Show how to prove the theorem in this case. (hint: define $C = \{x \in K \text{ s.t. } \langle y, x \rangle = 1\}$; show that C is a compact convex set and apply Minkowski's theorem for compact convex sets to C).*

2. *Use Exercise 2.1 to show that when K is a closed pointed convex cone, such a $y \in \mathbb{R}^n$ verifying $\langle y, x \rangle > 0$ for all $x \in K \setminus \{0\}$ exists.*

Exercise 2.3 (Examples of cones). *For each of the following sets: show that it is a closed convex pointed cone with nonempty interior, identify the extreme rays and give a simple expression for the dual cone:*

1. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \; \forall i = 1, \ldots, n\}$
2. $\mathbf{Q}^3 = \{(x_1, x_2, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ : \sqrt{x_1^2 + x_2^2} \leq t\}$
3. $K = \{(x, y, z) \in \mathbb{R}^2_+ \times \mathbb{R} : \sqrt{xy} \geq |z|\}$

Bonus question: Show that there is a linear invertible map $A : \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $A(\mathbf{Q}^3) = K$.