Exercises for revision class

Contents

1 Diagonally dominant matrices

A matrix $A \in \mathbf{S}^n$ is called *diagonally dominant* if $A_{ii} \geq \sum_{j \neq i} |A_{ij}|$ for all $i = 1, \ldots, n$. Let $\mathcal{D}_n \subset \mathbf{S}^n$ be the set of diagonally dominant matrices.

(a) Show that if A is diagonally dominant then it is positive semidefinite.

(b) Recall the definition of *proper cone*. Show that the set $\mathcal{D}_n$ is a proper cone in $\mathbf{S}^n$.

(c) Show that the extreme rays of $\mathcal{D}_n$ are spanned by the matrices

$$e_i e_i^T \quad (i = 1, \ldots, n) \quad \text{and} \quad (e_i \pm e_j)(e_i \pm e_j)^T \quad (1 \leq i < j \leq n).$$

where $e_i \in \mathbb{R}^n$ is the vector with 1 in the i'th component and 0 elsewhere.

2 Euclidean distance matrices

Let $(d_{ij})_{1 \leq i < j \leq n}$ be $\binom{n}{2}$ positive numbers. Show that the following two assertions are equivalent:

(i) There exist points $x_1, \ldots, x_n \in \mathbb{R}^k$ (for some k) such that $d_{ij} = \|x_i - x_j\|_2$ for all $i < j$.

(ii) The $n \times n$ symmetric matrix $D = \left[d_{ij}^2\right]_{1 \leq i,j \leq n}$ (where $d_{ii} = 0$) is negative semidefinite on the subspace orthogonal to $e = (1, \ldots, 1) \in \mathbb{R}^n$. [*We say that a matrix $A \in \mathbf{S}^n$ is negative semidefinite on a subspace L if $x^T A x \leq 0$ for all $x \in L$*]

3 Binary quadratic optimisation: Nesterov's $2/\pi$ result

Let A be a real symmetric matrix of size $n \times n$, and consider the following binary quadratic optimisation problem:

$$\text{maximise} \quad x^T A x \quad : \quad x \in \{-1, 1\}^n. \tag{1}$$

Let v^* be the optimal value of (1).

(a) Consider the semidefinite program:

$$\text{maximise} \quad \operatorname{Tr}(AX) \quad : \quad X \succeq 0 \text{ and } X_{ii} = 1, \ \forall i = 1, \ldots, n. \tag{2}$$

Let p^*_{SDP} be the optimal value of (2). Show that $v^* \leq p^*_{SDP}$.

From now on we are going to assume that A is positive semidefinite. The purpose of the rest of this problem is to show that $\frac{2}{\pi} p^*_{SDP} \leq v^*$. To prove this inequality, we will use a "randomised rounding" scheme similar to the one we saw in lecture for the maximum cut problem.

(b) Let X be the optimal solution (2) and let $v_1, \ldots, v_n \in \mathbb{R}^r$ with $r = \operatorname{rank}(X)$ such that $X_{ij} = \langle v_i, v_j \rangle$ for all $i, j = 1, \ldots, n$. Define the random variable $y \in \{-1, 1\}^n$ as follows:

$$y_i = \operatorname{sign}(\langle v_i, Z \rangle)$$

where Z is a standard Gaussian variable on $\mathbb{R}^r$. We saw in lecture that

$$\mathbb{E}[y_i y_j] = \frac{2}{\pi} \arcsin(X_{ij}) \quad \forall 1 \leq i, j \leq n,$$

which you can use without proof. Show that:

$$v^* \geq \mathbb{E}[y^T A y] = \frac{2}{\pi}\mathrm{Tr}(A \arcsin[X]).$$

where $\arcsin[X]$ is the matrix obtained by applying the arcsin function to each entry of X, i.e., $\arcsin[X]_{ij} = \arcsin(X_{ij})$.

(c) Recall the Schur product theorem:

> *Schur product theorem*: If $P \succeq 0$ and $Q \succeq 0$ then $P \odot Q \succeq 0$ where $P \odot Q$ is the entrywise product of P and Q.

Use the Schur product theorem (without proof) to show that if $X \succeq 0$ then $\arcsin[X] - X \succeq 0$. [*Hint: Use the fact that* $\arcsin(x) = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} x^{2k+1}$ *for* $x \in [-1,1]$].

(d) Using the positive semidefinite assumption on A show then that $\mathrm{Tr}(A \arcsin[X]) \geq \mathrm{Tr}(AX)$. Conclude that $v^* \geq \frac{2}{\pi} p^*_{SDP}$.

4 Lovász ϑ number

Let $G = (V, E)$ be a graph. Recall the definition of Lovász theta number from Lecture 10:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n, X \in \mathbf{S}^n}{\text{maximise}} & \sum_{i=1}^n x_i \\ \text{subject to} & X_{ii} = x_i \qquad i \in V \\ & X_{ij} = 0 \qquad ij \in E \\ & \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0 \end{array} \tag{3}$$

(a) Show that the dual of (3) can be expressed as

$$\begin{array}{ll} \text{minimise} & Z_{00} \\ \text{subject to} & z_i = (1 + Z_{ii})/2 \qquad \forall i \in V \\ & Z_{ij} = 0 \qquad \forall \{i,j\} \in \overline{E} \\ & \begin{bmatrix} Z_{00} & z^T \\ z & Z \end{bmatrix} \succeq 0 \end{array} \tag{4}$$

where $\overline{E} = \{\{i,j\} : i \neq j \text{ and } \{i,j\} \notin E\}$ is the complement of E [*Hint: you can use the fact (without proof) that* $\left[\begin{smallmatrix} A & B^T \\ B & C \end{smallmatrix}\right] \succeq 0 \iff \left[\begin{smallmatrix} A & -B^T \\ -B & C \end{smallmatrix}\right] \succeq 0$].

(b) Show that (4) can be simplified to:

$$\begin{array}{ll} \text{minimise} & Z_{00} \\ \text{subject to} & Z_{ii} = 1 \qquad \forall i \in V \\ & Z_{ij} = 0 \qquad ij \in \overline{E} \\ & \begin{bmatrix} Z_{00} & \mathbf{1}^T \\ \mathbf{1} & Z \end{bmatrix} \succeq 0 \end{array} \tag{5}$$

where $\mathbf{1}$ denotes the vector with all ones [*Hint: given* (z, Z) *feasible for* (4), *consider* $\tilde{Z}_{ij} = Z_{ij}/(z_i z_j)$].

(c) Use Slater condition to verify that (5) and (3) have the same optimal values.

(d) Show that for any graph G with n vertices we have $\vartheta(G)\vartheta(\bar{G}) \geq n$ where $\bar{G} = (V, \bar{E})$.

5 Faces of the positive semidefinite cone

(a) Recall the definition of a *face* of a convex set. Let V be a subspace of $\mathbb{R}^n$. Show that

$$F_V = \left\{ Y \in \mathbf{S}^n_+ : \operatorname{im} Y \subseteq V \right\}$$

is a face of the positive semidefinite cone. What is its dimension?

(b) Find a $C \in \mathbf{S}^n$ such that $\operatorname{argmin}_{X \in \mathbf{S}^n_+} \langle C, X \rangle = F_V$ (this shows that F_V is an *exposed face* of $\mathbf{S}^n_+$).

(c) Let $X \in \mathbf{S}^n_+$. Show that the smallest face of $\mathbf{S}^n_+$ containing X is $F_{\operatorname{im} X}$.

6 Existence of extreme points

Given a set $C \subseteq \mathbb{R}^n$ we say that C contains a *straight line* if there exists $x \in C$ and $v \in \mathbb{R}^n$ such that $x + tv \in C$ for all $t \in \mathbb{R}$.

(a) Let C be a nonempty closed convex set that does not contain any straight lines. Show that C has an extreme point [*Hint: you can use an argument by induction on the dimension of C, similar to the proof of Theorem 1.2 we did in lecture*].

(b) Conversely, show that if C is a closed convex set with an extreme point then it does not contain any straight lines.

7 Extreme points in linear programming

(a) Recall the definition of *extreme point* of a convex set.

(b) Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and consider the convex set $P = \{x \in \mathbb{R}^n_+ : Ax = b\}$. Show that any extreme point x of P satisfies $|\operatorname{supp}(x)| \leq m$ where $\operatorname{supp}(x) := \{i \in [n] : x_i \neq 0\}$ [*Hint: Show that if x is an extreme point of P then* $\ker(A) \cap \{y \in \mathbb{R}^n : \operatorname{supp}(y) \subseteq \operatorname{supp}(x)\} = \{0\}$].

Use Exercise 6 to show that if P is not empty then it has at least one extreme point.

(c) Use the result of part (b) to prove Carathéodory's theorem:

> *Carathéodory's theorem*: Let $S \subset \mathbb{R}^N$ be a finite set. Then any element of $\operatorname{conv}(S)$ can be expressed as a convex combination of at most $N + 1$ points of S.

8 Extreme points in semidefinite programming

Part (a) of this exercise is the analogue of Exercise 7(a) for the case of semidefinite programming.

(a) Let $\mathcal{A} : \mathbf{S}^n \to \mathbb{R}^m$ be a linear map, $b \in \mathbb{R}^m$ and let $C = \{X \in \mathbf{S}^n_+ : \mathcal{A}(X) = b\}$. Show that any extreme point X of C satisfies $r(r+1)/2 \leq m$ where $r = \operatorname{rank} X$ [*Hint: Show that if X is an extreme point of C then* $\ker(\mathcal{A}) \cap \{Y \in \mathbf{S}^n : \operatorname{im}(Y) \subseteq \operatorname{im}(X)\} = \{0\}$].

Use Exercise 6 to show that if C is nonempty then it has at least one extreme point.

(b) Let $A, B \in \mathbf{S}^n$. Use part (a) to show that the set

$$R(A,B) = \{(x^T A x, x^T B x) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2$$

is convex. (This set is known as the *numerical range* or *field of values* of the pair (A, B).) [*Hint: consider* $\{(\langle A, X\rangle, \langle B, X\rangle) : X \in \mathbf{S}^n_+\}$].

(c) Prove the following result, known as the *S-lemma*: Let $A, B \in \mathbf{S}^n$ and assume that for any $x \in \mathbb{R}^n$, $x^T A x \geq 0 \Rightarrow x^T B x \geq 0$. Assume furthermore that there exists $z \in \mathbb{R}^n$ such that $z^T A z > 0$. Show that there exists $\lambda \geq 0$ such that $B \succeq \lambda A$.

Give an example of $A, B \in \mathbf{S}^2$ to show that the condition of existence of $z \in \mathbb{R}^n$ such that $z^T A z > 0$ cannot be removed in general.

9 Matrix square root

(a) Let $A, B \succ 0$. Show that if $A^2 \succeq B^2$ then $A \succeq B$ [*Hint: let v be an eigenvector of $A - B$ and consider $v^T(A+B)(A-B)v$*].

(b) Give an example of $A, B \in \mathbf{S}^2_{++}$ such that $A \succeq B$ but $A^2 \not\succeq B^2$.

10 Some facts about nonnegative polynomials

(a) Show that if $p \in \mathbb{R}[x_1, \ldots, x_n]$ is nonnegative then it has even degree.

(b) Show that if $p = \sum_k q_k^2$ then necessarily $\deg q_k \leq (\deg p)/2$.

11 Homogeneous and nonhomogeneous polynomials

A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is called *homogeneous of degree d* if it only involves monomials of degree exactly d. Given a nonhomogeneous polynomial p of degree d we can *homogenise* it by introducing an additional variable x_0 via

$$\bar{p}(x_0, x_1, \ldots, x_n) = x_0^d p(x_1/x_0, \ldots, x_n/x_0) \tag{6}$$

(a) Show that (6) is well-defined. What is the homogenisation of $p(x_1, x_2) = x_1^2 x_2^2 - 2x_1 x_2 + 1$?

(b) Show that p is nonnegative if and only if $\bar{p}$ is nonnegative.

(c) Show that p is a sum of squares if and only if $\bar{p}$ is a sum of squares.

12 A nonnegative polynomial that is not a sum of squares

In lecture we saw the Motzkin polynomial $M(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ which is an explicit example of a nonnegative polynomial that is not a sum of squares in the case $(n, 2d) = (2, 6)$ (where n is the number of variables and $2d$ the degree). In this exercise we look at a polynomial in 3 variables of degree 4 (i.e., $(n, 2d) = (3, 4)$) that is nonnegative but not a sum-of-squares. Consider the following polynomial (due to Choi and Lam [CL77]).

$$Q(x,y,z) = x^2y^2 + x^2z^2 + y^2z^2 + 1 - 4xyz.$$

(a) Show that $Q(x,y,z) \geq 0$ for all $(x,y,z) \in \mathbb{R}^2$.

(b) Show that Q is not a sum of squares.

13 Positive and decomposable maps

(Based on exercise 3.178 in [BPT12]) A map $\Lambda : \mathbf{S}^{n_1} \to \mathbf{S}^{n_2}$ is called *positive* if $\Lambda(A) \succeq 0$ whenever $A \succeq 0$.

(a) Show that if Λ has the form $\Lambda(A) = \sum_{i=1}^{r} P_i^T A P_i$ where $P_1, \ldots, P_r \in \mathbb{R}^{n_1 \times n_2}$ then Λ is positive. Such maps are called *decomposable*.

(b) To any linear map $\Lambda : \mathbf{S}^{n_1} \to \mathbf{S}^{n_2}$ we can consider the polynomial $p(x, y) = y^T \Lambda(xx^T) y$ where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$. Show that Λ is a positive map if and only if p is nonnegative. Show that Λ is decomposable if and only if p is a sum-of-squares.

(c) Consider the following map $\Lambda : \mathbf{S}^3 \to \mathbf{S}^3$ due to M.-D. Choi [Cho75]:

$$\Lambda(A) = 2 \begin{bmatrix} a_{11} + a_{22} & 0 & 0 \\ 0 & a_{22} + a_{33} & 0 \\ 0 & 0 & a_{33} + a_{11} \end{bmatrix} - A.$$

(i) Show that Λ is positive [*Hint: in the case* $a_{33} \geq a_{11}$ *use* $\Lambda(A) = DAD + \begin{bmatrix} 2a_{22} & -2a_{12} & 0 \\ -2a_{12} & 2a_{33} & 0 \\ 0 & 0 & 2a_{11} \end{bmatrix}$ *with* $D = \operatorname{diag}(1, 1, -1)$*; then generalise using cyclic symmetry of* Λ].

(ii) Show that Λ is not decomposable. [*Hint: show that the associated polynomial* $p(x, y)$ *is not a sum-of-squares*].

14 Sum-of-squares on the hypercube

(Based on [Ble15]) Let $s(x) = x_1 + \cdots + x_n$.

(a) Show that the function $f(x) = (n - s(x))(n - 2 - s(x))$ is nonnegative on $\{-1, 1\}^n$.

(b) Show that f is not 1-sos on $\{-1, 1\}^n$.

(c) Show that f is 2-sos on $\{-1, 1\}^n$ [*Hint: what is* $(1 - x_i - x_j + x_i x_j)^2$ *?*]

References

[Ble15] G. Blekherman. Final homework in course "Real Algebraic Geometry and Optimization" at Georgia Tech, 2015. https://sites.google.com/site/grrigg/home/real-algebraic-geometry-and-optimization. 6

[BPT12] Grigoriy Blekherman, Pablo A. Parrilo, and Rekha R. Thomas. *Semidefinite optimization and convex algebraic geometry*. SIAM, 2012. 6

[Cho75] Man-Duen Choi. Positive semidefinite biquadratic forms. *Linear Algebra and its Applications*, 12(2):95–100, 1975. 6

[CL77] Man Duen Choi and Tsit Yuen Lam. An old question of Hilbert. *Queen's papers in pure and applied mathematics*, 46:385–405, 1977. 5