10 Nonnegative polynomials, sums of squares and semidefinite programming

Definition 10.1. We say a polynomial $p \in \mathbb{R}[x]$ is *nonnegative* if $p(x) \geq 0$ for all $x \in \mathbb{R}$.

Note that if we have a "good" way of checking nonnegativity of polynomials then we can also minimise (or maximise) polynomials. Indeed if $p(x)$ is a polynomial then

$$\min_{x\in\mathbb{R}} p(x) \quad = \quad \max \quad \gamma \quad \text{s.t.} \quad p - \gamma \text{ is nonnegative.} \tag{1}$$

It is not difficult to verify that if p is nonnegative then:

- The degree of p is even and the leading coefficient (i.e., the coefficient of x^{2d} if $\deg p = 2d$) is nonnegative.
- Any real root of p has even multiplicity.

These conditions can also be shown to be sufficient (see proof of Theorem 10.1 below).

Definition 10.2. We say that a polynomial $p \in \mathbb{R}[x]$ is a sum-of-squares if there exist polynomials $q_1, \ldots, q_k \in \mathbb{R}[x]$ such that $p = \sum_{i=1}^k q_i^2$.

It is clear that if p is a sum of squares, then it is nonnegative. The converse is also true for polynomials in one variable.

Theorem 10.1. *A univariate polynomial $p(x) = \sum_{k=0}^{2d} p_k x^k$ of degree $2d$ is nonnegative if and only if there exist q_1, q_2 of degree $\leq d$ such that $p = q_1^2 + q_2^2$.*

Proof. The implication $\Leftarrow$ is clear. Assume $p(x)$ is nonnegative. Since p has real coefficients, if $p(z) = 0$ then $p(\bar{z}) = 0$. Furthermore if z is a real root of p then it must have even multiplicity. This implies that we can write:

$$p(x) = p_{2d} \prod_{i=1}^{d} (x - z_i)(x - \bar{z}_i) = |q(x)|^2$$

where $q(x) = \sqrt{p_{2d}} \prod_{i=1}^d (x - z_i)$ (note that $p_{2d} \geq 0$ since p is nonnegative). If we let $q_1(x) = \mathrm{Re}[q(x)]$ and $q_2 = \mathrm{Im}[q(x)]$ (one can easily verify that these are polynomials of degree at most d) we get the desired result. $\square$

The next theorem shows that checking if a polynomial is a sum of squares can be decided using a semidefinite feasibility program:

Theorem 10.2. *A polynomial $p(x) = \sum_{k=0}^{2d} p_k x^k$ is a sum of squares if, and only if, there exists a positive semidefinite matrix M of size $(d+1) \times (d+1)$ such that*

$$p_k = \sum_{\substack{0 \leq i,j \leq d \\ i+j=k}} M_{ij} \qquad \forall k = 0, \ldots, 2d. \tag{2}$$

(The rows and columns of the matrix M are indexed by $0, \ldots, d$ instead of $1, \ldots, d+1$ for convenience.)

Proof. We first prove $\Rightarrow$: assume p is a sum of squares, i.e., $p = \sum_{i=1}^{\ell} q_i^2$ where $\deg q_i \leq d$. Denote by $[q_i] \in \mathbb{R}^{d+1}$ the vector of coefficients of q_i and $[x] = (1, x, \dots, x^d)$ so that $q_i(x) = [q_i]^T[x]$. Then we have

$$p(x) = \sum_{i=1}^{\ell} q_i(x)^2 = \sum_{i=1}^{\ell} [x]^T [q_i][q_i]^T [x] = [x]^T M [x]$$

where $M = \sum_{i=1}^{\ell} [q_i][q_i]^T$. Note that $M \succeq 0$. To see that the linear equations (2) hold, note that since $p(x) = [x]^T M[x]$ and

$$[x]^T M[x] = \sum_{0 \leq i,j \leq d} M_{ij} x^i x^j = \sum_{k=0}^{2d} \left(\sum_{\substack{0 \leq i,j \leq d \\ i+j=k}} M_{ij} \right) x^k \tag{3}$$

we must have (by matching coefficients) $p_k = \sum_{\substack{0 \leq i,j \leq d \\ i+j=k}} M_{ij}$ for all $k = 0, \dots, 2d$.

We now prove the converse. Assume $M \succeq 0$ satisfies (2). Then we can decompose M as $M = \sum_{i=1}^{\ell} [q_i][q_i]^T$ for some vectors $[q_i] \in \mathbb{R}^{d+1}$. Define $q_i(x) = [q_i]^T[x]$ where $[x] = (1, x, \dots, x^d)$. Then one can easily verify that $\sum_{i=1}^{\ell} q_i(x)^2 = [x]^T M[x] = p(x)$ where the last equality follows from (2) and by the same calculation as in (3). □

By combining Theorems 10.1 and 10.2 we can check if a polynomial in one variable is nonnegative using a semidefinite feasibility problem. In fact using (1) we can minimize any polynomial in one variable using semidefinite programming:

$$\begin{aligned}
\min_{x \in \mathbb{R}} p(x) &= \max \quad \gamma \quad \text{s.t.} \quad p - \gamma \text{ nonnegative} \\
&= \max_{\gamma \in \mathbb{R}, M \in \mathbf{S}^{d+1}} \quad \gamma \quad \text{s.t.} \quad \begin{cases} M \succeq 0 \\ p_k = \sum_{\substack{0 \leq i,j \leq d \\ i+j=k}} M_{ij} & \forall k = 1, \dots, 2d \\ p_0 - \gamma = M_{00} \end{cases}
\end{aligned}$$

Example 1 (Polynomials of degree 2)**.** *We know from high-school algebra that a polynomial* $p(x) = ax^2 + bx + c$ *is nonnegative iff* $b^2 - 4ac \leq 0$ *and* $a, c \geq 0$*. Theorems 10.1 and 10.2 tell us that this polynomial is nonnegative if and only if there exists a matrix* $M \in \mathbf{S}^2$ *such that*

$$\begin{aligned}
&M \succeq 0, \\
&M_{00} = c, \\
&M_{01} + M_{10} = b, \\
&M_{11} = a.
\end{aligned}$$

This is equivalent to saying that $\begin{bmatrix} c & b/2 \\ b/2 & a \end{bmatrix}$ *is positive semidefinite, which in turn is equivalent to having* $b^2 - 4ac \leq 0$ *and* $a, c \geq 0$*.*

Let P_{2d} be the cone of nonnegative polynomials (in one variable) of degree $2d$:

$$P_{2d} = \left\{ (p_0, \dots, p_{2d}) \in \mathbb{R}^{2d+1} \; : \; \sum_{k=0}^{2d} p_k x^k \geq 0 \;\; \forall x \in \mathbb{R} \right\}.$$

Theorem 10.3. *P_{2d} is a proper cone.*

Proof. We have to check that P_{2d} is closed, convex, pointed and has nonempty interior. Note that P_{2d} can be written as

$$P_{2d} = \bigcap_{x\in\mathbb{R}} \underbrace{\left\{(p_0,\ldots,p_{2d}) : \sum_{k=0}^{2d} p_k x^k \geq 0\right\}}_{H_x}$$

where each H_x is a closed halfspace. Thus P_{2d} is closed and convex as an intersection of closed convex sets. Checking that P_{2d} is pointed is easy. We leave it as an exercise to verify that the polynomial $x^{2d}+1$ is in the interior of P_{2d}. □