11 The cone of nonnegative univariate polynomials

Recall that $P_{2d}$ is the cone of nonnegative polynomials (in one variable) of degree $2d$:

$$P_{2d} = \left\{ (p_0, \ldots, p_{2d}) : \sum_{k=0}^{2d} p_k x^k \geq 0 \forall x \in \mathbb{R} \right\}. \quad (1)$$

We saw last time that $P_{2d}$ is a proper cone and that it has the following semidefinite representation:

$$p \in P_{2d} \iff \exists M \in \mathbb{S}_{++}^{d+1} \text{ s.t. } \sum_{0 \leq i,j \leq d \atop i+j=k} M_{ij} = p_k. \quad (2)$$

This means that any conic program over $P_{2d}$ is actually a semidefinite program.

**Duality** For any $x \in \mathbb{R}$ consider the vector $y_x \in \mathbb{R}^{2d+1}$ defined by:

$$y_x = (1, x, x^2, \ldots, x^{2d}) \in \mathbb{R}^{2d+1}. \quad (3)$$

Let $M_{2d}$ be the curve drawn by these vectors in $\mathbb{R}^{2d+1}$, known as the moment curve of degree $2d$:

$$M_{2d} = \{ y_x : x \in \mathbb{R} \}. \quad (4)$$

Observe that the definition (1) of $P_{2d}$ simply expresses that $P_{2d}$ is the dual cone of $M_{2d}$, i.e.,

$$P_{2d} = M_{2d}^\ast. \quad (5)$$

By the biduality theorem for closed convex cones (Theorem 2.3, cf. also footnote 1) we thus get automatically that

$$P_{2d}^* = \text{cl cone}(M_{2d}). \quad (5)$$

The vectors $y_x$, when interpreted as linear forms on the space of polynomials, correspond to point evaluations. Indeed if $p$ is a polynomial of degree $2d$ with coefficients $(p_0, \ldots, p_{2d})$, then the inner product $\langle p, y_x \rangle$ is nothing but $p(x)$, the point evaluation of $p$ at $x \in \mathbb{R}$. It is clear that point evaluations $y_x$ live in $P_{2d}^*$ (since the point evaluation of any nonnegative polynomial at $x$ gives a nonnegative number). Equation (5) tells us that (up to closure) any element in $P_{2d}^*$ is a nonnegative combination of point evaluations.

**Remark 1** (Remark on the closure in (5)). The cone generated by the moment curve $M_{2d}$ is not closed in general and so we cannot remove the closure operation in (5). For example one can verify $(0, 0, 1) \in \text{cl} \text{cone}(M_{2d}) \setminus \text{cone}(M_{2d})$: indeed, on the one hand it is not possible to write $(0, 0, 1)$ as a conic combination of the $\{ y_x : x \in \mathbb{R} \}$, and on the other hand we have $(0, 0, 1) = \lim_{x \to \infty} \frac{1}{x^2} y_x$. The main reason why $\text{cone}(M_{2d})$ is not closed is because we are allowing $x$ to be arbitrarily large on the real line. If we restrict $x$ in the definition of the moment curve (4) to live in a compact interval $x \in [a, b]$ then the cone would be closed in this case.

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1 In lecture 2 we only defined the dual of a cone; however the definition works for any set $S$: the dual of a set $S \subseteq \mathbb{R}^n$ is $\{ y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \forall x \in S \}$. Theorem 2.3 easily extends to show that for any set $S$, $S^{**} = \text{cl cone}(S)$. 

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Moment interpretation of $P^*_m$

Consider the following question, called the (truncated) moment problem: given numbers $(y_0, y_1, \ldots, y_{2d}) \in \mathbb{R}^{2d+1}$, does there exist a nonnegative measure $\mu$ on $\mathbb{R}$ such that $\int x^k d\mu(x) = y_k$ for all $k = 0, \ldots, 2d$? If the answer is yes we will say that $y$ is a valid moment vector. It is clear that not any vector $y \in \mathbb{R}^{2d+1}$ is a valid moment vector. For example we must have $y_k \geq 0$ for any $k$ even. Also we must have $y_2 + (y_0 - 2)y_1^2 \geq 0$ since $y_2 + (y_0 - 2)y_1^2 = \int (x - y)^2 d\mu(x) \geq 0$. What other inequalities must be true? If $p$ is any polynomial nonnegative on $\mathbb{R}$ then we must have $\int p(x) d\mu(x) \geq 0$. If we let $p = (p_0, \ldots, p_{2d})$ be the coefficients of this polynomial this means we must have:

$$0 \leq \int p(x) d\mu(x) = \sum_{k=0}^{2d} p_k x^k d\mu(x) = \sum_{k=0}^{2d} p_k \int x^k d\mu(x) = \sum_{k=0}^{2d} p_k y_k.$$ 

In other words if $y$ is a valid moment vector then we must have

$$\langle p, y \rangle \geq 0 \quad \forall p \in P_{2d}.$$ 

This means, by definition of dual cone, that $y \in P_{2d}^*$. Note that the vectors $y_x$ defined in (3) are actually valid moment vectors: $y_x$ is simply the moment vector for the Dirac probability measure $\delta_x$ that puts all its mass at $\{x\}$. Any conic combination of these vectors is a valid moment vector. Indeed if $y = \sum_{i=1}^N p_i y_{x_i}$ where $p_1, \ldots, p_N \geq 0$, then $y$ is the moment vector of the nonnegative atomic measure $\sum_{i=1}^N p_i \delta_{x_i}$. It thus follows that any element of $\text{conv}(y_x : x \in \mathbb{R})$ is a valid moment vector. To summarise we have the following duality picture:

$$\text{nonnegative polynomials of degree } \leq 2d \quad \longleftrightarrow \quad \text{moment vectors } (y_0, \ldots, y_{2d}) \text{ of nonnegative measures} \quad (\text{up to closure})$$

Let us try to push this duality picture further. We have seen that if $p$ is a polynomial of degree $2d$ then the minimum of $p$ over $\mathbb{R}$ can be expressed as:

$$\min_{x \in \mathbb{R}} p(x) = \max \gamma : p - \gamma \in P_{2d}.$$ 

The maximization problem on the right-hand side is a conic program over $P_{2d}$ that is strictly feasible. Let us try to write its dual. Let $y \in P_{2d}^*$ denote our dual variable for the constraint $p - \gamma \in P_{2d}$ which allows us to write $\langle p - \gamma, y \rangle \geq 0$, i.e., $\gamma y_0 \leq \langle p, y \rangle$. Since we are interested in the objective function $\gamma$, we want $y_0 \geq 1$ and so the dual problem becomes:

$$\min_y \langle p, y \rangle \quad \text{s.t.} \quad y \in P_{2d}^*, \; y_0 = 1.$$ 

(6)

We know that elements of $P_{2d}^*$ correspond (up to closure) to moments of nonnegative measures. Requiring that $y_0 = 1$ means we are restricting ourselves to probability measures. Thus problem (6) is equivalent to

$$\min \int p d\mu : \mu \text{ probability measure on } \mathbb{R}.$$ 

(7)

It is interesting to compare (7) to the problem $\min \{ p(x) : x \in \mathbb{R} \}$. It is not hard to see that the two have the same value: indeed let $p^* = \min_{x \in \mathbb{R}} p(x)$ and $x^*$ be a minimizer of $p$. Then clearly for any nonnegative probability measure on $\mathbb{R}$ we have $\int p d\mu(x) \geq \int p^* d\mu(x) = p^*$ and so the value of (7) is greater than or equal $p^*$. Now if we choose $\mu = \delta_{x^*}$ then the value of $\int p d\mu$ is equal to $p^*$. Thus this shows that the optimal value of (7) is indeed $p^*$. 

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Note that even though $p$ can be a complicated nonconvex polynomial, problem (7) has a linear objective function (in $\mu$), irrespective of what $p$ is. However (7) is an infinite-dimensional problem since the underlying space is the space of measures on $\mathbb{R}$. Note that the objective function of (7) only depends on the moments up to degree $2d$ of the measure $\mu$. Problem (6) can be seen as a finite-dimensional “projection” of (7) where we only work with the moments, up to degree $2d$, of these measures.