11 The cone of nonnegative univariate polynomials

Recall that P_{2d} is the cone of nonnegative polynomials (in one variable) of degree $2d$:

$$P_{2d} = \left\{ (p_0, \dots, p_{2d}) \; : \; \sum_{k=0}^{2d} p_k x^k \geq 0 \;\; \forall x \in \mathbb{R} \right\}. \tag{1}$$

We saw last time that P_{2d} is a proper cone and that it has the following *semidefinite representation*:

$$p \in P_{2d} \iff \exists M \in \mathbf{S}_+^{d+1} \text{ s.t. } \sum_{\substack{0 \leq i,j \leq d \\ i+j=k}} M_{ij} = p_k. \tag{2}$$

This means that any conic program over P_{2d} is actually a semidefinite program.

Duality For any $x \in \mathbb{R}$ consider the vector $y_x \in \mathbb{R}^{2d+1}$ defined by:

$$y_x = (1, x, x^2, \dots, x^{2d}) \in \mathbb{R}^{2d+1}. \tag{3}$$

Let M_{2d} be the curve drawn by these vectors in $\mathbb{R}^{2d+1}$, known as the *moment curve* of degree $2d$:

$$M_{2d} = \{ y_x : x \in \mathbb{R} \}. \tag{4}$$

Observe that the definition (1) of P_{2d} simply expresses that P_{2d} is the dual cone[1] of M_{2d}, i.e.,

$$P_{2d} = M_{2d}^*.$$

By the biduality theorem for closed convex cones (Theorem 2.3, cf. also footnote 1) we thus get automatically that

$$P_{2d}^* = \operatorname{cl}\operatorname{cone}(M_{2d}). \tag{5}$$

The vectors y_x, when interpreted as linear forms on the space of polynomials, correspond to *point evaluations*. Indeed if p is a polynomial of degree $2d$ with coefficients $(p_0, \dots, p_{2d})$, then the inner product $\langle p, y_x \rangle$ is nothing but $p(x)$, the *point evaluation* of p at $x \in \mathbb{R}$. It is clear that point evaluations y_x live in P_{2d}^* (since the point evaluation of any nonnegative polynomial at x gives a nonnegative number). Equation (5) tells us that (up to closure) any element in P_{2d}^* is a nonnegative combination of point evaluations.

Remark 1 (Remark on the closure in (5))**.** *The cone generated by the moment curve M_{2d} is not closed in general and so we cannot remove the closure operation in (5). For example one can verify $(0,0,1) \in \operatorname{cl}(\operatorname{cone}(M_2)) \setminus \operatorname{cone}(M_2)$: indeed, on the one hand it is not possible to write $(0,0,1)$ as a conic combination of the $\{y_x : x \in \mathbb{R}\}$, and on the other hand we have $(0,0,1) = \lim_{x \to \infty} \frac{1}{x^2} y_x$. The main reason why $\operatorname{cone}(M_{2d})$ is not closed is because we are allowing x to be arbitrarily large on the real line. If we restrict x in the definition of the moment curve (4) to live in a compact interval $x \in [a,b]$ then the cone would be closed in this case.*

[1] In lecture 2 we only defined the dual of a *cone*; however the definition works for any set S: the dual of a set $S \subseteq \mathbb{R}^n$ is $\{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \; \forall x \in S\}$. Theorem 2.3 easily extends to show that for any set S, $S^{**} = \operatorname{cl}\operatorname{cone}(S)$.

Moment interpretation of P_{2d}^* Consider the following question, called the (truncated) *moment problem*: given numbers $(y_0, y_1, \dots, y_{2d}) \in \mathbb{R}^{2d+1}$, does there exist a nonnegative measure μ on $\mathbb{R}$ such that $\int x^k d\mu(x) = y_k$ for all $k = 0, \dots, 2d$? If the answer is yes we will say that y is a *valid moment vector*. It is clear that not any vector $y \in \mathbb{R}^{2d+1}$ is a valid moment vector. For example we must have $y_k \geq 0$ for any k even. Also we must have $y_2 + (y_0 - 2)y_1^2 \geq 0$ since $y_2 + (y_0 - 2)y_1^2 = \int (x - y_1)^2 d\mu(x) \geq 0$. What other inequalities must be true? If p is any polynomial nonnegative on $\mathbb{R}$ then we must have $\int p(x) d\mu(x) \geq 0$. If we let $p = (p_0, \dots, p_{2d})$ be the coefficients of this polynomial this means we must have:

$$0 \leq \int p(x) d\mu(x) = \int \sum_{k=0}^{2d} p_k x^k d\mu(x) = \sum_{k=0}^{2d} p_k \int x^k d\mu(x) = \sum_{k=0}^{2d} p_k y_k.$$

In other words if y is a valid moment vector then we must have

$$\langle p, y \rangle \geq 0 \quad \forall p \in P_{2d}.$$

This means, by definition of dual cone, that $y \in P_{2d}^*$. Note that the vectors y_x defined in (3) are actually valid moment vectors: y_x is simply the moment vector for the Dirac probability measure δ_x that puts all its mass at $\{x\}$. Any conic combination of these vectors is a valid moment vector. Indeed if $y = \sum_{i=1}^N p_i y_{x_i}$ where $p_1, \dots, p_N \geq 0$, then y is the moment vector of the nonnegative *atomic measure* $\sum_{i=1}^N p_i \delta_{x_i}$. It thus follows that any element of $\mathrm{conv}(y_x : x \in \mathbb{R})$ is a valid moment vector. To summarise we have the following duality picture:

nonnegative polynomials of degree $\leq 2d$	$\xleftrightarrow{\textit{duality}}$	moment vectors $(y_0, \dots, y_{2d})$ of nonnegative measures (up to closure)

Let us try to push this duality picture further. We have seen that if p is a polynomial of degree $2d$ then the minimum of p over $\mathbb{R}$ can be expressed as:

$$\min_{x \in \mathbb{R}} p(x) \quad = \quad \max \ \gamma \ \ : \ \ p - \gamma \in P_{2d}.$$

The maximization problem on the right-hand side is a conic program over P_{2d} that is strictly feasible. Let us try to write its dual. Let $y \in P_{2d}^*$ denote our dual variable for the constraint $p - \gamma \in P_{2d}$ which allows us to write $\langle p - \gamma, y \rangle \geq 0$, i.e., $\gamma y_0 \leq \langle p, y \rangle$. Since we are interested in the objective function γ, we want $y_0 = 1$ and so the dual problem becomes:

$$\min_y \quad \langle p, y \rangle \quad \text{s.t.} \quad y \in P_{2d}^*, \ y_0 = 1. \tag{6}$$

We know that elements of P_{2d}^* correspond (up to closure) to moments of nonnegative measures. Requiring that $y_0 = 1$ means we are restricting ourselves to *probability* measures. Thus problem (6) is equivalent to

$$\min \int p d\mu \quad : \quad \mu \text{ probability measure on } \mathbb{R}. \tag{7}$$

It is interesting to compare (7) to the problem $\min\{p(x) : x \in \mathbb{R}\}$. It is not hard to see that the two have the same value: indeed let $p^* = \min_{x \in \mathbb{R}} p(x)$ and x^* be a minimizer of p. Then clearly for any nonnegative probability measure on $\mathbb{R}$ we have $\int p d\mu(x) \geq \int p^* d\mu(x) = p^*$ and so the value of (7) is greater than or equal p^*. Now if we choose $\mu = \delta_{x^*}$ then the value of $\int p d\mu$ is equal to p^*. Thus this shows that the optimal value of (7) is indeed p^*.

Note that even though p can be a complicated nonconvex polynomial, problem (7) has a linear objective function (in μ), irrespective of what p is. However (7) is an infinite-dimensional problem since the underlying space is the space of measures on $\mathbb{R}$. Note that the objective function of (7) only depends on the moments up to degree $2d$ of the measure μ. Problem (6) can be seen as a finite-dimensional "projection" of (7) where we only work with the moments, up to degree $2d$, of these measures.