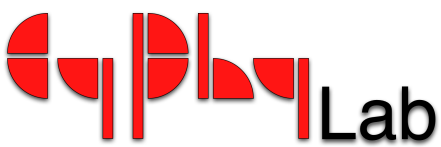
Secure state-estimation for dynamical systems under active adversaries

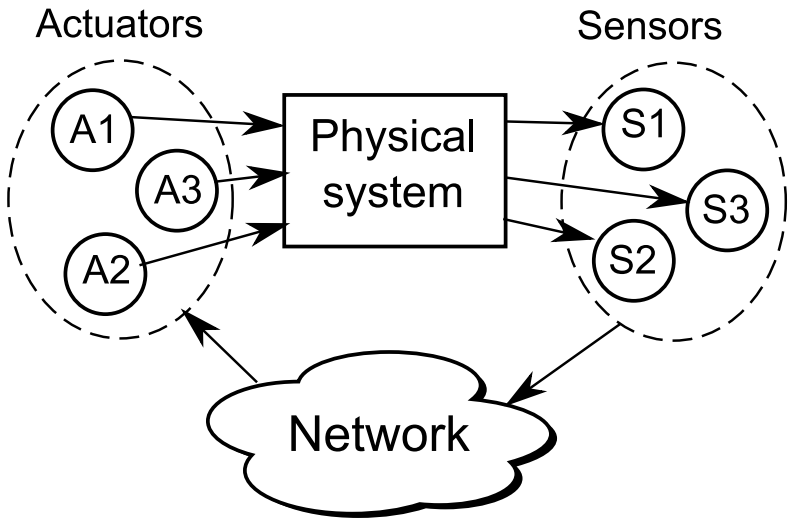
Hamza Fawzi

Joint work with Paulo Tabuada and Suhas Diggavi



Why security for control systems?

- Control systems are *physical* processes (chemical plants, power grid, mechanical system, etc.)



- Control systems becoming larger (large sensor networks) and increasingly open to the *cyber*-world (e.g., internet) $\Rightarrow$ **increased vulnerability to attacks**
- Examples of real attacks: Sewage control system (Queensland, Australia, 2000), Natural gas pipelines (Russia, 2000), Stuxnet (2010), ...
- **Need efficient ways to detect attacks on control systems...**

For more info on security for control systems see *[Cardenas, Amin, Sastry, 2008]*

Security for control systems

- (Some of the) existing works on adversarial, malicious attacks:
  - Optimal control in the presence of intelligent jammer (cf. Gupta, Langbort and Basar, 2010)
    - game-theoretic approach; attacker's objective is to maximize cost function
  - Secure state-estimation for power network against malicious attacks (cf. Pasqualetti, Dorfler, Bullo (2011))
    - attack-detection filter is proposed, but computationally expensive (combinatorial, test all possible attack sets)
- **This talk: efficient algorithm to estimate the state of a linear dynamical system when sensors are attacked**

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- Physical process modeled as a linear dynamical system

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- **Objective: Given observations $y^{(0)}, \ldots, y^{(T-1)}$: recover state $x^{(0)}$ of physical plant from observations (attack set K is unknown)**

Error correction

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- Note:

  can correct $q = 0$ errors $\equiv$ can recover $x^{(0)}$ from $(Cx^{(0)}, \ldots, CA^{T-1}x^{(0)}) \equiv (A, C)$ observable

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  - Example of a very bad pair (A, C): $A =$ identity, $C =$ identity (easy to see that even $q = 1$ does not satisfy condition above: take x to be supported on one component)

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- $\Rightarrow$ $q = 1$ **error is NOT correctable in this case**

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- One little problem: It is NP-hard... :-(

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- Define ℓ_0 norm of $E^{(T)}$ as the number of nonzero rows of $E^{(T)}$ ($=$ number of attacked sensors):

$$\|E^{(T)}\|_{\ell_0} = |\text{rowsupport}(E^{(T)})|$$

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  - Collect observations from $t = 0$ to $t = T - 1$ in a $p \times T$ matrix:

$$\underbrace{\left[\begin{array}{c|c|c} y^{(0)} & \dots & y^{(T-1)} \end{array}\right]}_{Y^{(T)} \in \mathbf{R}^{p \times T}} = \underbrace{\left[\begin{array}{c|c|c} Cx & \dots & CA^{T-1}x \end{array}\right]}_{\Phi^{(T)} x} + \underbrace{\left[\begin{array}{c|c|c} e^{(0)} & \dots & e^{(T-1)} \end{array}\right]}_{E^{(T)} \in \mathbf{R}^{p \times T}}$$

- Define ℓ_0 norm of $E^{(T)}$ as the number of nonzero rows of $E^{(T)}$ (= number of attacked sensors):

$$\|E^{(T)}\|_{\ell_0} = |\text{rowsupport}(E^{(T)})|$$

- Let's rewrite the previous "*unbeatable*" decoder using these notations:
  - smallest number of attacked sensors that explain the received observations:

$$\underset{x}{\text{minimize}} \; \| \underbrace{Y^{(T)} - \Phi^{(T)} x}_{E^{(T)}} \|_{\ell_0}$$

The "ℓ_1" relaxation

- ℓ_0 decoder: NP-hard

$$\underset{x}{\text{minimize}} \; \| Y^{(T)} - \Phi^{(T)} x \|_{\ell_0}$$

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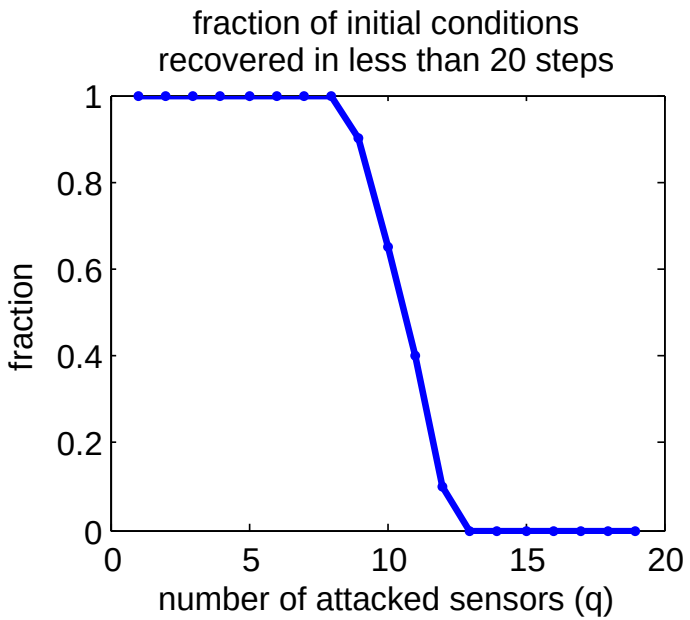
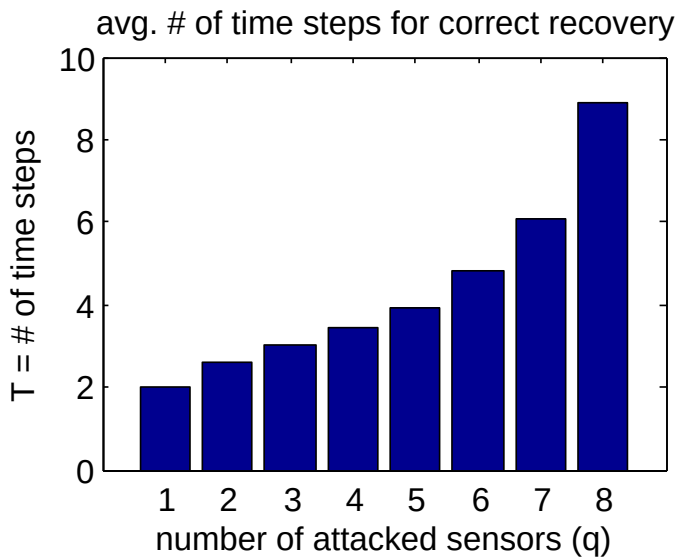
- $\ell_0 \to \ell_1$ relaxation idea used in compressed sensing (recovery of sparse signals), and error correction over the reals (cf. Candes, Tao, Donoho, etc.)

Numerical example 1

- Randomly generated system (A, C) with $n = 30$ and $p = 20$ (Gaussian entries)
- Used ℓ_1/ℓ_2 decoder



(a)



(b)

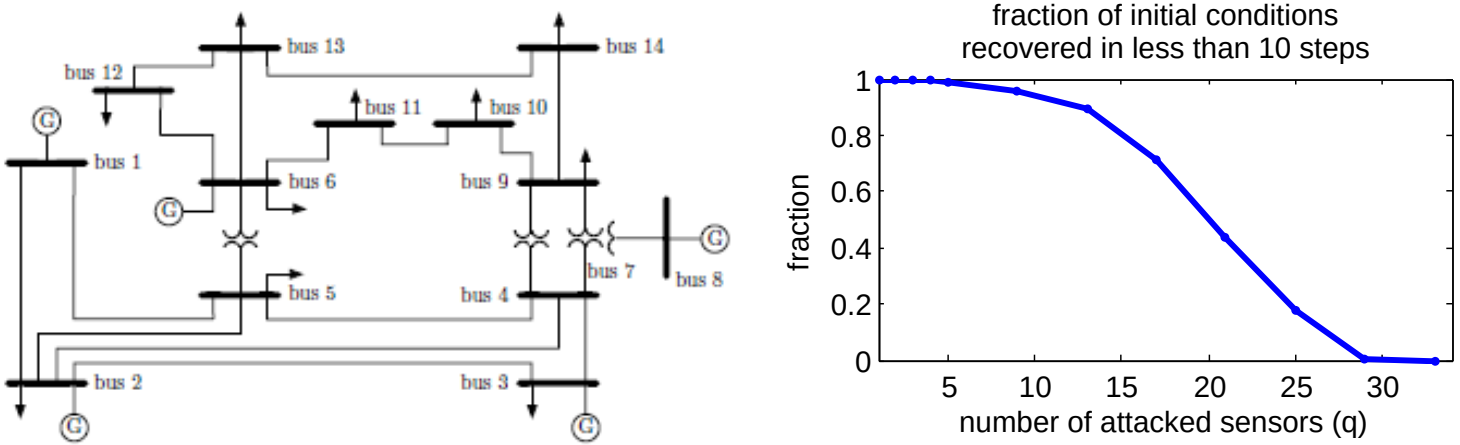
Figure: (a) Fraction of initial conditions (out of 20) that were correctly recovered in less than $T = 20$ time steps, for different values of q. (b) Average number of time steps it took to correctly recover the initial state, as a function of the number of corrupted components.

Numerical example 2

Electric power network: IEEE 14-bus power network (5 generators, 14 buses)

- $n = 2 \times 5 = 10$ states for the rotor angles δ_i and the frequencies $d\delta_i/dt$ of each generator i
- $p = 35$ sensors to measure: real power injections at every bus (14 sensors), real power flows along every branch (20 sensors), rotor angle at generator 1 (1 sensor) [1]

Used ℓ_1/ℓ_∞ decoder



(a) **(b)**

Figure: (a) IEEE 14-bus power network (b) Fraction of initial conditions that were correctly recovered in less than $T = 10$ steps. For each value of q, 200 simulations were carried out with different initial conditions and different sets of attacked sensors.

[1] cf. [Pasqualetti, Dorfler, Bullo 2010]. Thanks to Fabio Pasqualetti from UCSB for the data!

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(recall that $\Phi x = \left[\begin{array}{c|c|c} Cx & \cdots & CA^{T-1}x \end{array}\right]$)

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- Question: Given (A, C) how to check above condition? no known efficient way...

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Thank you!

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