Equivariant semidefinite lifts of regular polygons

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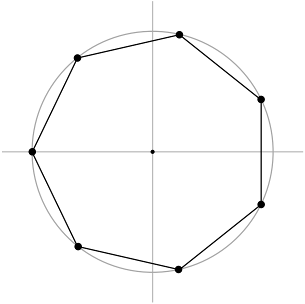
Equivariant psd lifts and regular polygons

Definition

Let P be a polytope in $\mathbb{R}^n$ and assume that P is invariant under action of G. A psd lift $P = \pi(\mathbf{S}_+^d \cap L)$ is called G-equivariant if there exists a homomorphism $\rho : G \to GL_d(\mathbb{R})$ such that:

- $\rho(g)Y\rho(g)^T \in L$ for all $Y \in L$ and $g \in G$.
- $\pi(\rho(g)Y\rho(g)^T) = g\pi(Y)$ for all $Y \in \mathbf{S}_+^d \cap L$ and $g \in G$.

Symmetry group of regular N-gon is *dihedral group* (order $2N$) and consists of N rotations and N reflections.



Previous work on regular polygons

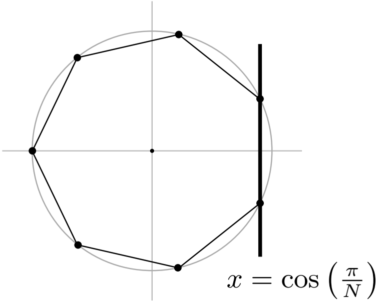
$$N = 2^n\text{-gon}$$

	Equivariant	Non-equivariant
LP	Lower bound: 2^n [GPT13] Upper bound: 2^n (trivial)	Lower bound: n [Goe14] Upper bound: $2n + 1$ (Ben-Tal & Nemirovski)
SDP	Lower bound: $(\ln 2)(n-1)$ Upper bound: $2n-1$	Lower bound: $\Omega\left(\sqrt{\frac{n}{\log n}}\right)$ [GPT13] Upper bound: $2n-1$

Constructing equivariant psd lifts

Facet inequality for regular N-gon:

$$\ell(x, y) := \cos(\pi/N) - x \geq 0.$$



Main concern in this talk is to find *sum-of-squares certificates* for this inequality, i.e., find polynomials $f_i \in \mathbb{R}[x, y]$ such that:

$$\ell = \sum_i f_i^2$$

on *the vertices of the N-gon*.

The space of functions on the vertices of the N-gon

Let $\mathcal{F}(N,\mathbb{R})$ the space of functions on the vertices of the N-gon.

- Any function on the vertices of the N-gon can be regarded as a polynomial:

$$\mathcal{F}(N,\mathbb{R}) \cong \mathbb{R}[x,y]/I$$

  where I is the vanishing ideal of the vertices of the N-gon.

- $\mathcal{F}(N,\mathbb{R})$ decomposes according to *degree* of polynomials:

$$\mathcal{F}(N,\mathbb{R}) = \mathrm{TPol}_0(N) \oplus \mathrm{TPol}_1(N) \oplus \cdots \oplus \mathrm{TPol}_{\lfloor N/2 \rfloor}(N)$$

- Each $\mathrm{TPol}_k(N)$ is spanned by $\{c_k, s_k\}$ where

$$c_k(x,y) = \mathrm{Re}[(x+iy)^k] \quad s_k(x,y) = \mathrm{Im}[(x+iy)^k].$$

- A decomposition of $f \in \mathcal{F}(N,\mathbb{R})$ into the basis $\{c_k, s_k\}$ is a real discrete Fourier decomposition.

Sum-of-squares hierarchy

Facet inequality for regular N-gon:

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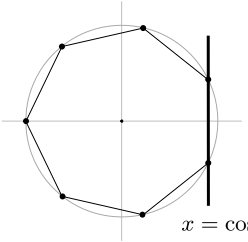


- Sum-of-squares hierarchy at level d is exact if there exist functions $f_i \in \mathrm{TPol}_0(N) \oplus \cdots \oplus \mathrm{TPol}_d(N)$ such that:

  $$\ell = \sum_i f_i^2$$

  where equality is understood in $\mathcal{F}(N, \mathbb{R})$.

- In this case we have an equivariant psd lift of the regular N-gon of size

  $$\dim(\mathrm{TPol}_0(N) \oplus \cdots \oplus \mathrm{TPol}_d(N)) = 2d - 1.$$

Sparse sum-of-squares certificates

- One can potentially obtain equivariant lifts that are smaller than the sum-of-squares hierarchy:
- Assume there is a set $K \subset \{0, \ldots, \lfloor N/2 \rfloor\}$ such that we can write:

$$\ell = \sum_i f_i^2$$

where each $f_i \in \bigoplus_{k \in K} \mathrm{TPol}_k(N)$. Then this automatically yields an equivariant psd lift of the regular N-gon of size $\dim V \leq 2|K|$
$\Rightarrow$ if K is sparse, this lift can be much smaller than the lift produced by the hierarchy.

Main results

1. The sum-of-squares hierarchy requires exactly $\lceil N/4 \rceil$ levels.
2. There exists an equivariant psd lift of the regular 2^n-gon of size $2n - 1$.
3. Any equivariant psd lift of the regular N-gon has size at least $\ln(N/2)$.

Lasserre/sum-of-squares hierarchy

Proposition

The sum-of-squares hierarchy of the regular N-gon requires at least $N/4$ iterations.

Proof.

(Due to G. Blekherman) Assume we can write:

$$\ell(x, y) = SOS(x, y) + g(x, y) \tag{1}$$

where $g(x, y)$ is a polynomial that vanishes on the vertices of the regular N-gon. Since $g \neq 0$ and $g(\cos\theta, \sin\theta)$ has N roots on the unit circle, we have $\deg g \geq N/2$. Thus we get that $\deg SOS = \deg(\ell - g) \geq N/2$. □

One can show that $\lceil N/4 \rceil$ iterations are enough.

An equivariant psd lift for the regular 2^n-gon of size $2n-1$

Theorem

Let $\ell = \cos(\pi/2^n) - x \in \mathcal{F}(2^n, \mathbb{R})$. *Then* ℓ *admits a sum-of-squares certificate with frequencies in*

$$K = \{0\} \cup \{2^i, i = 0, \ldots, n-2\}.$$

More precisely, there exist functions $h_k \in \mathrm{TPol}_0(2^n) \oplus \mathrm{TPol}_{2^k}(2^n)$ *for* $k = 0, 1, \ldots, n-2$ *such that:*

$$\ell = \sum_{k=0}^{n-2} h_k^2. \tag{2}$$

Proof by induction

For integer N, let $\ell_N = \cos(\pi/N) - x \in \mathcal{F}(N, \mathbb{R})$.

Lemma

If $\ell_N \in \mathcal{F}(N, \mathbb{R})$ has sos certificate with frequencies in K, then $\ell_{2N} \in \mathcal{F}(2N, \mathbb{R})$ has sos certificate with frequencies in $\{0, 1\} \cup 2K$.

Proof.

- Trigonometric identity, true for all $\theta \in \mathbb{R}$:

$$\cos\left(\frac{\pi}{2N}\right) - \cos\theta = \alpha_N\left(\cos\left(\frac{\pi}{N}\right) - \cos(2\theta)\right) + 2\alpha_N(\cos(\pi/(2N)) - \cos(\theta))^2$$

  where

$$\alpha_N = \frac{\sin(\pi/(2N))}{2\sin(\pi/N)} \geq 0$$

- Note: if $\ell_N = \cos(\pi/N) - \cos\theta \in \mathcal{F}(N, \mathbb{R})$ has sos certificate with frequencies in K, then $\cos(\pi/N) - \cos(2\theta) \in \mathcal{F}(2N, \mathbb{R})$ has sos certificate with frequencies in $2K$.
- Thus $\ell_{2N} = \cos(\pi/(2N)) - \cos\theta \in \mathcal{F}(2N, \mathbb{R})$ has sos certificate with frequencies in $\{0, 1\} \cup 2K$.

Explicit sum-of-squares certificate

$$\frac{\cos\left(\frac{\pi}{2^n}\right) - \cos(\theta)}{\sin\left(\frac{\pi}{2^n}\right)} = \sum_{k=0}^{n-2} \frac{\left(\cos\left(2^k \cdot \frac{\pi}{2^n}\right) - \cos(2^k\theta)\right)^2}{2^k \sin\left(2^{k+1} \cdot \frac{\pi}{2^n}\right)} \bmod I$$

An equivariant $(\mathbf{S}^3_+)^{n-1}$ lift of the 2^n-gon

Theorem

The regular 2^n-gon is the set of points $(x_0, y_0) \in \mathbb{R}^2$ such that there exist real numbers $x_1, y_1, \ldots, x_{n-2}, y_{n-2}, y_{n-1}$ satisfying:

$$\begin{bmatrix} 1 & x_{k-1} & y_{k-1} \\ x_{k-1} & \frac{1+x_k}{2} & \frac{y_k}{2} \\ y_{k-1} & \frac{y_k}{2} & \frac{1-x_k}{2} \end{bmatrix} \succeq 0 \text{ for } k = 1, \ldots, n-2 \text{ and } \begin{bmatrix} 1 & x_{n-2} & y_{n-2} \\ x_{n-2} & \frac{1}{2} & \frac{y_{n-1}}{2} \\ y_{n-2} & \frac{y_{n-1}}{2} & \frac{1}{2} \end{bmatrix} \succeq 0.$$

An $\mathbf{S}^{2n-1}$-lift

Can also write the lift with a single block, using the moment matrix for the subspace

$$V = \mathrm{TPol}_0 \oplus \bigoplus_{k=0}^{n-2} \mathrm{TPol}_{2^k}.$$

Example: The regular 16-gon is the set of $(u_1, v_1) \in \mathbb{R}^2$ for which the following matrix is psd (for some $u_2, \dots, u_6, v_2, \dots, v_8$):

$$\begin{bmatrix} 2 & 2u_1 & 2v_1 & 2u_2 & 2v_2 & 2u_4 & 2v_4 \\ 2u_1 & 1+u_2 & v_2 & u_1+u_3 & v_1+v_3 & u_3+u_5 & v_3+v_5 \\ 2v_1 & v_2 & 1-u_2 & -v_1+v_3 & u_1-u_3 & -v_3+v_5 & u_3-u_5 \\ 2u_2 & u_1+u_3 & -v_1+v_3 & 1+u_4 & v_4 & u_2+u_6 & v_2+v_6 \\ 2v_2 & v_1+v_3 & u_1-u_3 & v_4 & 1-u_4 & -v_2+v_6 & u_2-u_6 \\ 2u_4 & u_3+u_5 & -v_3+v_5 & u_2+u_6 & -v_2+v_6 & 1 & v_8 \\ 2v_4 & v_3+v_5 & u_3-u_5 & v_2+v_6 & u_2-u_6 & v_8 & 1 \end{bmatrix} \succeq 0$$

Lower bound on equivariant lifts

- More convenient to work with Hermitian sum-of-squares (instead of real). Let $\mathcal{F}(N,\mathbb{C})$ be the space of complex-valued functions on the vertices of the N-gon. Then $\mathcal{F}(N,\mathbb{C})$ decomposes into:

$$\mathcal{F}(N) = \bigoplus_{k\in\mathbb{Z}_N} \mathbb{C}e_k \quad \text{where} \quad e_k(\theta) = e^{-ik\pi/N}e^{ik\theta}.$$

- We say that $h \in \mathcal{F}(N,\mathbb{C})$ is supported on $K \subseteq \mathbb{Z}_N$ if h is a linear combination of $\{e_k, k \in K\}$.

Theorem (Structure theorem)

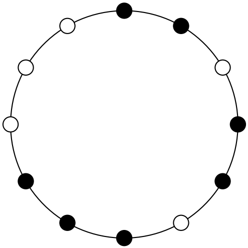
If the regular N-gon has an equivariant Hermitian psd lift of size d then there exists a set $K \subseteq \mathbb{Z}_N$ with $|K| = d$ and functions h_i supported on K s.t.:

$$\ell = \sum_i |h_i|^2$$

where $\ell := \cos(\pi/N) - x \in \mathcal{F}(N,\mathbb{C})$.

SOS-valid sets

- We call a set $K \subseteq \mathbb{Z}_N$ sos-valid if there exist functions h_i supported on K such that
$$\ell = \sum_i |h_i|^2.$$
- If K is sos-valid, then $K + \alpha$ is also sos-valid.
- Useful to represent sets K as a subset of the nodes of the cycle graph.



- Define the *in-diameter* of a set K to be the smallest integer r such that K is included in an interval $[x, x + r]$ for some $x \in \mathbb{Z}_N$.

A necessary condition for sos-valid sets

Main lemma giving necessary conditions for a set K to be SOS-valid:

Theorem

Let N be an integer and let $K \subseteq \mathbb{Z}_N$ be a set of frequencies. Assume that K can be decomposed into disjoint clusters $(C_\alpha)_{\alpha \in A}$:

$$K = \bigcup_{\alpha \in A} C_\alpha,$$

such that the following holds for some $1 \leq \gamma < N/2$:

(i) *For any $\alpha \in A$, C_α has in-diameter $\leq \gamma$.*

(ii) *For any $\alpha \neq \alpha'$, $d(C_\alpha, C_{\alpha'}) > \gamma$.*

Then the set K is not sos-valid (i.e., it is not possible to write the linear function ℓ as a sum of squares of functions supported on K).

Proof

Proof works by exhibiting a certain dual certificate. Define $\mathcal{L} : \mathcal{F}(N, \mathbb{C}) \to \mathbb{C}$ by:

$$\mathcal{L}(e_k) = \begin{cases} e^{-\frac{i\pi}{N}(k \text{ mod } N)} & \text{if } d(0,k) \leq \gamma \\ 0 & \text{else.} \end{cases} \tag{3}$$

where, for $k \in \mathbb{Z}_N$, k mod N is the unique element in

$$\left\{-\lceil N/2 \rceil + 1, \ldots, \lfloor N/2 \rfloor\right\}$$

that is equal to k modulo N. Then show that:

1. $\mathcal{L}(\ell) < 0$
2. $\mathcal{L}(|h|^2) \geq 0$ for any h supported on K.

Clustering

To finish the proof we show that if $K \subseteq \mathbb{Z}_N$ is small enough then it admits a valid clustering of the form considered in the previous lemma. One can prove:

Theorem

If $K \subseteq \mathbb{Z}_N$ has size $|K| \leq \ln(N/2)$ then K admits a valid clustering.

Proof uses greedy algorithm to construct clustering.

Conclusion

- Constructed an equivariant psd lift of the regular 2^n-gon of size $2n-1$.
  - Exponentially smaller than the lift of the sum-of-squares hierarchy, and exponentially smaller than equivariant LP lifts.
  - Main idea was to look for sparse sum-of-squares certificates. This idea could be useful in other applications to obtain smaller equivariant psd lifts.
- Proved matching lower bound on equivariant psd lifts for regular N-gons.
- Open question: What about non-equivariant psd lifts? Current lower bound (from quantifier elimination theory) is $\Omega(\sqrt{\frac{\log N}{\log\log N}})$.

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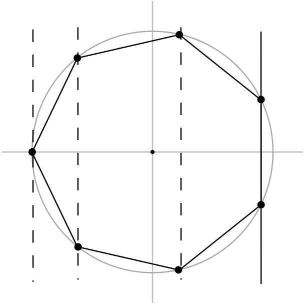
Thank you!

k-level polytopes

A polytope P is called k-level if any facet defining function $l(x) \geq 0$ takes at most k different values on the vertices of P.

Regular N-gon is $\lceil N/2 \rceil$-level:



Sum-of-squares hierarchy for k-level polytopes

Theorem (GT12)

If P is k-level then the $k-1$-level of the SOS-hierarchy for P is exact.

Proof.

Let $l(x) \geq 0$ be a facet linear inequality and let $0 = a_0 < \cdots < a_{k-1}$ be the k values taken by l on the vertices of P. Let q be a univariate polynomial with $\deg q = k-1$ such that $q(a_i) = \sqrt{a_i}$ and let $p = q^2$. Then, on the vertices of P, we have:

$$l(x) = p(l(x)) = q(l(x))^2.$$

Thus any facet functional l is $k-1$-sos modulo the vertices of P, i.e., the the $k-1$'st level of the hierarchy is exact. □

Key idea: Find a *globally nonnegative* univariate polynomial p such that $p(a_i) = a_i$ for $i = 0, \ldots, k-1$. Lagrange interpolation yields polynomial p with $\deg p = 2(k-1)$. Can we do better?

Nonnegative interpolation degree

Definition

A sequence $0 = a_0 < a_1 < \cdots < a_{k-1}$ has nonnegative interpolation degree d if there exists a *globally nonnegative* univariate polynomial p such that $p(a_i) = a_i$ for all $i = 0, \ldots, k-1$.

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Proposition

Let P be a k-level polytope in $\mathbb{R}^n$. Assume that for any facet-defining linear functional ℓ of P, the k values taken by ℓ on the vertices of P have nonnegative interpolation degree d. Then the $d/2$-iteration of the sum-of-squares hierarchy for P is exact (note that d is necessarily even).

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A characterization of sequences of length k with nonnegative interpolation degree k:

Proposition

Let $0 = a_0 < a_1 < \cdots < a_{k-1}$ be a sequence of length k. Let $q(x) = (x - a_0)\ldots(x - a_{k-1})$. The following are equivalent:

(i) *The sequence (a_i) has nonnegative interpolation degree k.*

(ii) *$q(x) \geq q'(0)x$ for $x \in \mathbb{R}$.*

Regular polygons

- For regular N-gon, the levels are

$$a_i = \cos(\pi/N) - \cos((2i+1)\pi/N), \quad i = 0, \ldots, \lceil N/2 \rceil - 1.$$

- The polynomial

$$q(x) = \prod_{i=0}^{\lceil N/2 \rceil - 1} (x - a_i)$$

  is nothing but a Chebyshev polynomial (when N is even).
- Using properties of Chebyshev polynomials, can show that $q(x) \geq q'(0)x$ (when N is a multiple of four).
- Thus the sequence (a_i) has nonnegative interpolation degree $\lceil N/2 \rceil$ and the sos hierarchy needs only $\lceil N/4 \rceil$ levels.