Equivariant semidefinite lifts and sum-of-squares hierarchies

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Lifts of convex sets

- Central question in optimization is to optimize a linear function ℓ on a convex set C:

$$\min_{x \in C} \ell(x).$$

- Need "good" description of C to solve optimization problem efficiently.
- Conic programming descriptions

Semidefinite programming representations

Semidefinite programming

$$\min\ \mathcal{L}(Y) \quad \text{subject to} \quad Y \in \mathbf{S}^d_+, \quad Y \in L$$

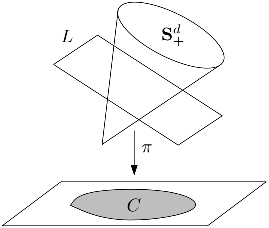
where L affine subspace of $\mathbf{S}^d$.

Positive semidefinite lift of C:

$$C = \pi(\mathbf{S}^d_+ \cap L)$$

where

- π linear map
- L affine subspace of $\mathbf{S}^d$

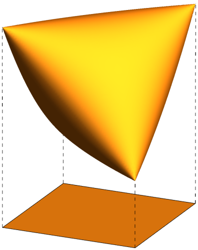


Consequence: Optimizing linear function over C is SDP of size d.

$$\min_{x \in C} \ell(x) \quad = \quad \min\ (\ell \circ \pi)(Y) \ \text{ s.t. } Y \in \mathbf{S}^d_+ \cap L$$

Example of psd lift

$$[-1,1]^2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \exists u \in \mathbb{R} \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & u \\ x_2 & u & 1 \end{bmatrix} \succeq 0 \right\} \qquad (1)$$



Sum-of-squares lifts

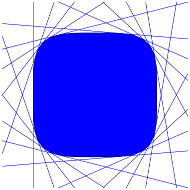
- Let $X \subseteq \mathbb{R}^n$. Goal: find a psd lift of $P = \mathrm{conv}(X)$.
- Let $\mathcal{F}(X, \mathbb{R})$ be the space of real-valued functions on X.

Theorem (Lasserre, Gouveia et al.)

Assume there is a subspace V of $\mathcal{F}(X, \mathbb{R})$ such that any facet-defining inequality $\ell(x) \leq b$ for $\mathrm{conv}(X)$ can be certified using sum-of-squares in V, i.e., there exist $f_1, \ldots, f_J \in V$:

$$b - \ell(x) = \sum_{j=1}^{J} f_j(x)^2 \quad \forall x \in X.$$

Then $\mathrm{conv}(X)$ has an (explicit) psd lift of size $\dim V$.



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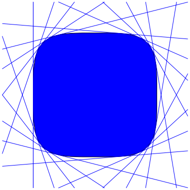


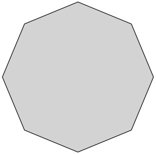
- Lasserre / theta-body hierarchy: take $V = \mathrm{Pol}_{\leq k}(X)$, subspace of polynomials of degree at most k on X.

Psd lifts that respect symmetry

- Assume that $X \subset \mathbb{R}^n$ is invariant under action of group G.

  We are interested in finding a psd lift of $P = \text{conv}(X)$ that **respects the** G**-symmetry of** P.

- Example of polytopes with symmetries:
  - Regular N-gon in $\mathbb{R}^2$
  - Hypercube $[-1, 1]^n$
  - Combinatorial polytopes: cut polytope, matching polytope, etc.



Equivariant psd lifts: definition

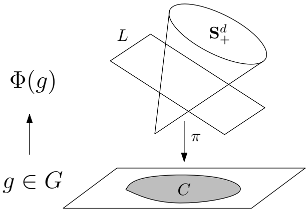
PSD lift

$$C = \pi(\mathbf{S}_+^d \cap L)$$

respects G-symmetry of C if any transformation $g \in G$ can be lifted to a transformation $\Phi(g)$ upstairs such that

$$\pi(\Phi(g)Y) = g\pi(Y)$$

for all $g \in G$, $Y \in \mathbf{S}_+^d \cap L$.



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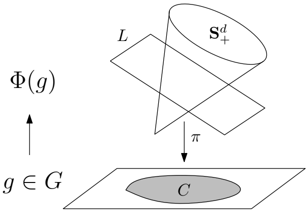

Requirements on Φ:

- $\Phi(g)$ acts on $\mathbf{S}^d$ by *congruence transformation*, i.e.,

$$\Phi(g)Y = \rho(g)Y\rho(g)^T$$

  where $\rho : G \to GL_d(\mathbb{R})$ homomorphism

- $\Phi(g)$ leaves L invariant

(These two conditions imply that $\Phi(g)$ leaves $\mathbf{S}^d_+ \cap L$ invariant)

A psd lift that respects G-symmetry of P is called equivariant psd lift

Sum-of-squares lifts that are **equivariant**

- Action of G on X induces action on $\mathcal{F}(X, \mathbb{R})$:

$$(g \cdot f)(x) = f(g^{-1}x)$$

- Subspace $V \subset \mathcal{F}(X, \mathbb{R})$ is called *G-invariant* if $G \cdot V = V$.

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Theorem (Sum-of-squares lifts that are **equivariant**)

*Assume there is a G-**invariant** subspace V of $\mathcal{F}(X, \mathbb{R})$ such that any facet-defining inequality $\ell(x) \leq b$ for $\operatorname{conv}(X)$ can be certified using sum-of-squares in V, i.e.,*

$$b - \ell(x) = \sum_{j=1}^{J} f_j(x)^2 \quad \textit{where} \quad f_j \in V \ \ (j = 1, \ldots, J).$$

*Then $\operatorname{conv}(X)$ has an (explicit) G-**equivariant** psd lift of size $\dim V$.*

- Lasserre/theta-body lifts are equivariant because $V = \mathrm{Pol}_{\leq k}(X)$ is G-invariant.

Main result: structure theorem

- Let $G \subset GL_n(\mathbb{R})$ and $X = G \cdot x_0$ where $x_0 \in \mathbb{R}^n$, and
$$P = \mathrm{conv}(G \cdot x_0).$$

Theorem (Structure theorem)

*Assume that P has a G-**equivariant** psd lift of size d.
Then there exists an G-**invariant** subspace V of $\mathcal{F}(X, \mathbb{R})$ with $\dim V \leq d^3$, such that any facet-defining inequality $\ell(x) \leq b$ for P has a sum-of-squares certificate with functions from V, i.e.,*

$$b - \ell(x) = \sum_{j=1}^{J} f_j(x)^2 \quad \textit{where} \quad f_j \in V \ \ (j = 1, \ldots, J).$$

Main result: structure theorem

Consequence of structure theorem: To study G-equivariant psd lifts of $\operatorname{conv}(X)$, need to study G-invariant subspaces of $\mathcal{F}(X, \mathbb{R})$.

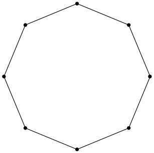
In particular we will be interested in decomposing $\mathcal{F}(X, \mathbb{R})$ into a direct sum of irreducibles:

$$\mathcal{F}(X, \mathbb{R}) = \bigoplus_i V_i$$

where each V_i is irreducible.
(Recall that subspace V is irreducible if it does not contain a nontrivial invariant subspace).

Example 1: regular polygons

$$X = \left\{ (\cos(2\pi k/N), \sin(2\pi k/N)) : k = 1, \dots, N \right\}$$



- $\mathcal{F}(X, \mathbb{R})$ decomposes as

$$\mathcal{F}(X, \mathbb{R}) = \mathrm{TPol}_0 \oplus \mathrm{TPol}_1 \oplus \cdots \oplus \mathrm{TPol}_{\lfloor N/2 \rfloor}$$

where $\mathrm{TPol}_k = \mathrm{span}(c_k, s_k)$

$$\begin{cases} c_k(x_1, x_2) = \mathrm{Re}[(x_1 + ix_2)^k] \\ s_k(x_1, x_2) = \mathrm{Im}[(x_1 + ix_2)^k] \end{cases}$$

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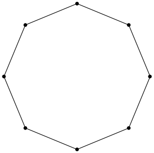

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Fact

Any G-invariant subspace V of $\mathcal{F}(X, \mathbb{R})$ has the form $V = \bigoplus_{k \in K} \mathrm{TPol}_k$ where $K \subseteq \{0, \ldots, \lfloor N/2 \rfloor\}$.

Example 1: regular polygons (continued)

$$\mathcal{F}(X,\mathbb{R}) = \mathrm{TPol}_0 \oplus \mathrm{TPol}_1 \oplus \cdots \oplus \mathrm{TPol}_{\lfloor N/2 \rfloor}$$

Theorem

Assume $K \subseteq \{0,\ldots,\lfloor N/2\rfloor\}$ *is such that we can write:*

$$\cos(\pi/N) - x_1 = \sum_{j=1}^{J} f_j(x)^2 \quad \forall x \in X. \tag{2}$$

where $f_1,\ldots,f_J \in \bigoplus_{k\in K} \mathrm{TPol}_k$*. Then* $|K| \geq \ln(N/2)/2$*.*

Consequence: any equivariant psd lift of the regular N-gon must have size at least $\Omega(\log N)$. Bound is tight (cf. James's talk tomorrow).

Example 2: parity polytope

- Parity polytope:

$$\mathrm{PAR}_n = \mathrm{conv}(\mathrm{EVEN}_n)$$

where

$$\mathrm{EVEN}_n = \left\{ x \in \{-1, +1\}^n : \prod_{i=1}^{n} x_i = 1 \right\}.$$

- PAR_n is symmetric with respect to:
  1. Switching the sign of an even number of components (G_{even} symmetry)
  2. Permutation of components ($\mathfrak{S}_n$ symmetry).

  Symmetry group of PAR_n

$$\begin{aligned} \Gamma &= \textit{evenly signed permutations} \\ &= G_{\mathrm{even}} \rtimes \mathfrak{S}_n. \end{aligned}$$

Example 2: parity polytope (continued)

Lemma (1)

- *$\mathcal{F}(EVEN_n, \mathbb{R})$ decomposes into Γ-irreducibles as:*

$$\mathcal{F}(EVEN_n, \mathbb{R}) = \mathrm{Pol}_0 \oplus \mathrm{Pol}_1 \oplus \cdots \oplus \mathrm{Pol}_{\lfloor n/2 \rfloor}\,.$$

where Pol_k is the space of square-free polynomials of degree at most k.

$$\dim \mathrm{Pol}_k = \begin{cases} \binom{n}{k} & \text{if } k < n/2 \\ \frac{1}{2}\binom{n}{n/2} & \text{if } k = n/2. \end{cases}$$

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Corollary (Structure theorem + Lemma 1 + Lemma 2)

Any Γ-equivariant psd lift for PAR_n has size at least $\binom{n}{\lceil n/4 \rceil}$.

Conclusion

- **Main message**: to study equivariant psd lifts, need to study invariant subspaces of $\mathcal{F}(X, \mathbb{R})$.

- *Lower bounds*
  - $\Omega(\log N)$ lower bound on equivariant psd lifts of regular N-gon.
  - Exponential lower bound on equivariant psd lifts of parity polytope, *via* lower bound on Lasserre/theta-body hierarchy. Similar analysis also gives exponential lower bound for equivariant psd lifts of cut polytope.

- *Upper bounds:* Understanding invariant subspaces of $\mathcal{F}(X, \mathbb{R})$ is also useful for constructions/upper bounds $\rightarrow$ *sparse sum-of-squares certificates* (see James's talk tomorrow FB04).

Papers: `arXiv:1312.6662` (parity polytope + cut polytope)
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