Separable states, semidefinite programming, and sums of squares

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Separable states

$$\mathtt{Sep}(n,m) = \mathrm{conv}\left\{ \ (x \otimes y)(x \otimes y)^\dagger \ \ : x \in \mathbb{C}^n, y \in \mathbb{C}^m \ \right\}.$$

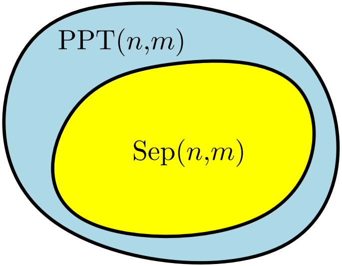
- $x^\dagger = \bar{x}^T$
- Full-dim convex cone in Herm$(nm) \simeq \mathbb{C}^{n^2 m^2}$
- $\mathtt{Sep}$ = set of *non-entangled* bipartite states on $\mathbb{C}^n \otimes \mathbb{C}^m$

PPT relaxation (positive partial transpose)

With T = transpose map, let

$$\texttt{PPT}(n, m) = \{\rho \in \mathsf{Herm}(nm) : \rho \geq 0 \text{ and } (I \otimes \mathrm{T})(\rho) \geq 0\}$$

(Check that $\texttt{Sep} \subset \texttt{PPT}$: $(I \otimes \mathrm{T})(xx^\dagger \otimes yy^\dagger) = xx^\dagger \otimes \bar{y}\bar{y}^\dagger \geq 0$)



Størmer–Woronowicz [60/70's]: $\texttt{Sep}(n, m) = \texttt{PPT}(n, m)$ iff $n + m \leq 5$

Semidefinite programming

A semidefinite program is an optimization program of the form:

$$\max_{x \in \mathbb{R}^N} \quad c^T x \quad \text{s.t.} \quad \mathcal{A}(x) \geq 0$$

where $\mathcal{A}(x) = A_0 + x_1 A_1 + \cdots + x_N A_N$.

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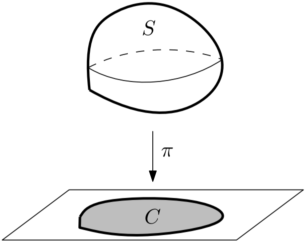
Linear optimization on PPT is a semidefinite program

SDP representation We say that C has a SDP representation if we can write

$$C = \pi(S)$$

where $S = \{x : \mathcal{A}(x) \geq 0\}$ and π is a linear map.



Lifting

- Can show that
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- If C has a SDP representation, then optimizing a linear function on C is a semidefinite program:
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- Lifting is very helpful from a complexity perspective

Other lifting examples

- Permutahedron

$$\operatorname{conv}\{(\sigma(1),\ldots,\sigma(n)) : \sigma \in S_n\}$$

has $n!$ vertices and $\sim 2^n$ facets. Can express it as the projection of the convex polytope of doubly stochastic matrices

$$DS_n = \{M \in \mathbb{R}^{n\times n} : M_{ij} \geq 0\ \forall ij \text{ and } M\mathbf{1} = \mathbf{1}^T M = 1\}$$

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- For perfect graphs Lovász showed

$$STAB(G) = \left\{ x \in \mathbb{R}^n : \exists X \text{ s.t. } \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0,\; X_{ii} = x_i, X_{ij} = 0 \; \forall ij \in E \right\}.$$

SDP lifts

Which convex sets C have an SDP lift? A necessary condition is that C is *semialgebraic* (Tarski)

Semialgebraic geometry

- A set is *semialgebraic* if it is a boolean combination (union, intersection, complement) of sets defined using polynomials equalities and inequalities
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- Scheiderer (2016): there are (many) convex semialgebraic sets that do not have an SDP representation

Main result

Theorem (Fawzi)

If $\mathcal{Sep}(n, m) \neq \mathcal{PPT}(n, m)$ then $\mathcal{Sep}(n, m)$ has no SDP lift. In other words, $\mathcal{Sep}(3, 3)$ and $\mathcal{Sep}(4, 2)$ have no SDP lift.

- Horodecki's formulation of $\mathtt{Sep}(n, m)$:

  $$\mathtt{Sep}(n, m) = \{\rho \in \mathrm{Herm}(nm) : (I \otimes \Phi)(\rho) \geq 0 \;\; \forall \Phi : \mathrm{M}_m \to \mathrm{M}_n \text{ positive}\}.$$

  Skowronek (2016) showed that for $\mathtt{Sep}(3, 3)$ it is not possible to reduce the quantifier $\forall\Phi$ to a finite number of maps $\Phi_1, \ldots, \Phi_k$.

- Result also includes as a special that the DPS (Doherty-Parrilo-Spedalieri) hierarchy does not converge in a finite number of levels when $n + m > 5$.

Duality, polynomials

$$\mathtt{Sep}(n,m) = \operatorname{conv}\left\{ \ (x \otimes y)(x \otimes y)^\dagger \ \ : x \in \mathbb{C}^n, y \in \mathbb{C}^m \ \right\}.$$

- Dual of $\mathtt{Sep}$: $\mathtt{Sep}^* \stackrel{def}{=} \{M \in \mathrm{Herm}(nm) : \langle M, \rho \rangle \geq 0 \ \forall \rho \in \mathtt{Sep}\}$.

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- Linear form nonnegative on $\mathtt{Sep}(n,m)$:

$$\underbrace{\langle M,(x\otimes y)(x\otimes y)^\dagger\rangle}_{\sum_{ijkl} M_{ij,kl} x_i \bar{x}_k y_j \bar{y}_l} \ \geq \ 0 \quad \forall (x,y)\in\mathbb{C}^n\times\mathbb{C}^m.$$

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- $\mathtt{Sep}^*$ = cone of nonnegative Hermitian biquadratic polynomials
- *Hermitian polynomial*: $f(z,\bar{z})$ polynomial in $(z,\bar{z})$ such that $f(z,\bar{z})\in\mathbb{R}$ for all $z\in\mathbb{C}^N$

$$f(z,\bar{z}) = \sum_{\alpha,\beta} f_{\alpha\beta} z^\alpha \bar{z}^\beta, \qquad f_{\alpha\beta} = \overline{f_{\beta\alpha}}$$

Sums of squares

- Hermitian polynomial $f(z, \bar{z})$ is a sum of squares if

$$f(z, \bar{z}) = \sum_i g_i(z, \bar{z})^2$$

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- $f(z,\bar{z})$ is a (complex) sum of squares if

$$f(z,\bar{z}) = \sum_i |g_i(z)|^2$$

for some (arbitrary) polynomials $g_i(z) \in \mathbb{C}[z]$

The two notions are different.

Sos relaxation of $\mathtt{Sep}$

$$p_M(x, \bar{x}, y, \bar{y}) = \sum_{ijkl} M_{ijkl} x_i \bar{x}_k y_j \bar{y}_l$$

- $\mathtt{Sep}^* = \{M \in \mathrm{Herm}(nm) : p_M$ is nonnegative$\}$.

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$\mathtt{Sep}$	$\subset$	$\mathtt{PPT}$
$\updownarrow$ (duality)		$\updownarrow$ (duality)
Nonnegative polynomials p_M	$\supset$	Sum-of-squares polynomials p_M

(For more details about duality between Sep/PPT and nonnegativity/sums of squares, see paper arXiv:1908.05155 joint with Kun Fang)

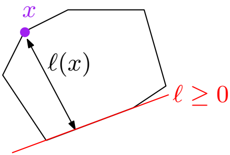
Semidefinite lifts

- Definition of SDP lift is hard to work with. Need a more algebraic way of thinking about it

- Given convex cone $C = \text{conv}(X)$, we associate a *slack matrix* $\boldsymbol{S}$ (potentially infinite) defined as follows:

$$\boldsymbol{S}(x, \ell) = \ell(x) \geq 0 \qquad \forall x \in X, \ell \in C^*$$

- If C polytope, then slack matrix $\boldsymbol{S}$ has size $\#$*vertices* $\times$ $\#$*facets*



Factorization theorem

$$\text{SDP lift of } C \quad \Leftrightarrow \quad \text{Factorization of } \boldsymbol{S}$$

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Theorem (Gouveia, Parrilo, Thomas)

$C = \text{conv}(X)$ has an SDP lift of size N iff one can find maps $A : X \to Herm_+^N$ and $B : C^* \to Herm_+^N$ such that we have the factorization

$$\boldsymbol{S}(x,\ell) = \text{Tr}[A(x)B(\ell)] \qquad \forall x \in X, \ell \in C^*$$

Generalizes a result of Yannakakis 1991 (LPs) to SDPs

SDP lifts and sums of squares

A corollary of the previous theorem is

Theorem

Assume $C = \text{conv}(X)$ *has an SDP lift of size* N*. Then there is a subspace* $\mathcal{V}$ *of functions on* X *of dimension at most* N^2 *s.t. for any* $\ell \in C^*$

$$\ell|_X = \sum_k h_k^2 \qquad \textit{where} \quad h_k \in \mathcal{V}.$$

Remark:

- Taking $\mathcal{V} = \text{Pol}_{\leq k}(X)$ space of polynomials of degree at most k gives the *Lasserre* hierarchy for $\text{conv}(X)$

General result in the real case

Theorem (Main, real case)

Let $p \in \mathbb{R}[\boldsymbol{x}]$ *be a nonnegative polynomial that is not sos. Let*

$$A = \{\alpha \in \mathbb{N}^n : \alpha \leq \beta \text{ for some } \beta \in \mathrm{support}(p)\}$$

be the "staircase" under support(p). *Then*

$$C_A = \mathrm{conv}\left\{(\boldsymbol{x}^\alpha)_{\alpha \in A} : \boldsymbol{x} \in \mathbb{R}^n\right\}$$

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- Application: Take $p =$ Motzkin (inhomogeneous) polynomial. Associated A is $\subseteq \left\{\alpha \in \mathbb{N}^2 : |\alpha| \leq 6\right\}$. Shows that $P_{2,6}^*$ has no SDP representation (where $P_{2,6}$ is set of nonneg. polynomials in 2 vars. of degree ≤ 6)

Proof

$$C_A = \text{conv}\left\{(x^\alpha)_{\alpha\in A} : x \in \mathbb{R}^n\right\}$$

- Linear functions nonnegative on $C_A \leftrightarrow$ nonnegative polynomials supported on A

Characterization of SDP lifts using sum-of-squares:

Theorem

C_A has an SDP representation iff there are functions $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, k)$ such that any nonnegative polynomial supported on A can be written as a sum of squares of functions from $\text{span}(f_1, \ldots, f_k)$.

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- **Semialgebraic functions are tame:** They are smooth (C^∞) almost everywhere (except on a set of measure 0)

Proof of main theorem

p nonnegative polynomial not sos, $A =$ staircase under support(p)

$$C_A = \text{conv}\left\{(x^\alpha)_{\alpha\in A} : x \in \mathbb{R}^n\right\}.$$

- Assume C_A has an SDP representation, and let $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be the semialgebraic functions associated to this representation
- Since the $(f_i)_{i=1,\ldots,k}$ are smooth almost everywhere, there is a point $a \in \mathbb{R}^n$ such that the f_i are all smooth at a
- Since A is the staircase under support(p), the polynomial $p(x + a)$ is supported on A, and since it is nonnegative, it must be a sum-of-squares from span($f_1, \ldots, f_k$). Shifting by a, this means that p is a sum of squares from span($\tilde{f}_1, \ldots, \tilde{f}_k$) where $\tilde{f}_i(x) = f_i(x - a)$

Smooth sums of squares

Proposition

Assume p is a homogeneous polynomial such that $p = \sum_j f_j^2$ for some arbitrary functions f_j that are C^∞ at the origin. Then p is a sum of squares of polynomials.

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- Proves theorem when p is homogeneous
- Additional technical argument based on Puiseux expansions is needed for general p

Main result, complex case

Theorem (Main, complex case)

Let p be a nonnegative Hermitian polynomial that is not sos. Let

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- If $\mathtt{Sep}(n, m) \neq \mathtt{PPT}(n, m)$, apply theorem above with $p =$ (dehomogenized) nonnegative Hermitian biquadratic on (n, m) variables that is not sos
- For $\mathtt{Sep}(3, 3)$ use the Choi polynomial. For $\mathtt{Sep}(4, 2)$ use a polynomial exhibited by Woronowicz and further studied by Ha and Kye.

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arXiv:1905.02575