Sparse sum-of-squares certificates on finite abelian groups

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Lifts of convex sets

- Central question in optimization is to optimize a linear function ℓ on a convex set C:

$$\min_{x \in C} \ell(x).$$

- Need “good” description of C to solve optimization problem efficiently.
- Conic programming descriptions

Semidefinite programming representations

Semidefinite programming

$$\min\ \mathcal{L}(Y) \quad \text{subject to} \quad Y \in \mathbf{S}^d_+, \quad Y \in L$$

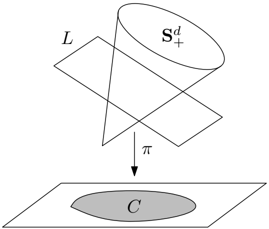
where L affine subspace of $\mathbf{S}^d$.

Positive semidefinite lift of C:

$$C = \pi(\mathbf{S}^d_+ \cap L)$$

where

- π linear map
- L affine subspace of $\mathbf{S}^d$

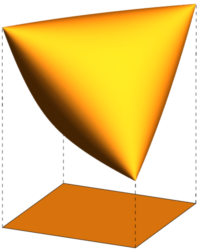


Consequence: Optimizing linear function over C is SDP of size d.

$$\min_{x \in C} \ell(x) \quad = \quad \min\ (\ell \circ \pi)(Y) \ \text{ s.t. } Y \in \mathbf{S}^d_+ \cap L$$

Example of psd lift

$$[-1,1]^2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \exists u \in \mathbb{R} \;\; \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & u \\ x_2 & u & 1 \end{bmatrix} \succeq 0 \right\} \qquad (1)$$



Sum-of-squares lifts

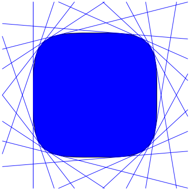
- Let $X \subseteq \mathbb{R}^n$. Goal: find a psd lift of $P = \text{conv}(X)$.
- Let $\mathcal{F}(X, \mathbb{C})$ be the space of complex-valued functions on X.

Theorem (Lasserre, Gouveia et al.)

Assume there is a subspace V of $\mathcal{F}(X, \mathbb{C})$ such that any affine function ℓ that is nonnegative on X, can be certified using sum-of-squares in V, i.e., there exist $f_1, \ldots, f_J \in V$:

$$\ell(x) = \sum_{j=1}^{J} |f_j(x)|^2 \quad \forall x \in X.$$

Then conv(X) *has an (explicit) Hermitian psd lift of size* dim V.



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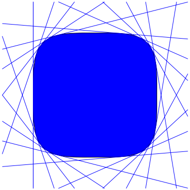


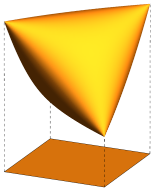
- Lasserre / theta-body hierarchy: take $V = \text{Pol}_{\leq k}(X)$, subspace of polynomials of degree at most k on X.

Example

- $X = \{-1, 1\}^2$ and let $V = \text{Pol}_{\leq 1}(X) = \text{span}(1, x_1, x_2)$.

- Can verify:

$$\forall (x_1, x_2) \in X : \quad \begin{cases} 1 - x_1 = \frac{1}{2}(1 - x_1)^2 \\ 1 - x_2 = \frac{1}{2}(1 - x_2)^2 \\ 1 + x_1 = \frac{1}{2}(1 + x_1)^2 \\ 1 + x_2 = \frac{1}{2}(1 + x_2)^2 \end{cases}$$



- Thus, $\text{conv}(X) = [-1, 1]^2$ has a psd lift of size $\dim V = 3$.

Moment polytopes

- Let G be a finite abelian group.
- A **character** χ of G is a homomorphism $G \to \mathbb{C}^*$, i.e.,

$$\chi(xy) = \chi(x)\chi(y) \quad \forall x, y \in G.$$

  - Example: $G = \{-1, 1\}^n$, if $S \subseteq [n]$ then $\chi(x) = \prod_{i \in S} x_i$ is a character.

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- **Key facts about characters:**
  1. The set of characters forms a *group*, denoted $\widehat{G}$ (dual group).
  2. The set of characters forms an orthonormal basis for $\mathcal{F}(G, \mathbb{C})$ for the inner product:

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g).$$

  Fourier decomposition of a function $f : G \to \mathbb{C}$:

$$f(x) = \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\chi(x) \quad \forall x \in G$$

Moment polytopes

Moment polytope Let $\mathcal{S} \subseteq \widehat{G}$. Define

$$\mathcal{M}(G, \mathcal{S}) = \text{conv}\Big\{(\chi(x))_{\chi \in \mathcal{S}} : x \in G\Big\} \subset \mathbb{C}^{|\mathcal{S}|}$$

Goal Construct psd lifts of $\mathcal{M}(G, \mathcal{S})$ using sum-of-squares.

Moment polytopes: example 1 (hypercube)

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Hypercube Let $G = \{-1,1\}^n$. Characters are of the form:

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- $\mathcal{S} = \{\chi_S : |S| = 2\}$:

$$\mathcal{M}(G, \mathcal{S}) = \text{conv}\Big\{(x_i x_j)_{i<j} : x \in \{-1, 1\}^n\Big\} \stackrel{\text{def}}{=} \text{CUT}_n$$

Moment polytopes: example 2 (cyclic group)

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Cyclic group Let $G = (\mathbb{Z}_N, +)$. Characters are of the form

$$\chi_k(x) = e^{2i\pi kx/N} \quad \text{where } k \in \{0, 1, \ldots, N-1\}$$

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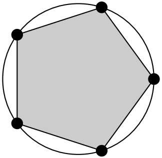
- $\mathcal{S} = \{\chi_1, \chi_{N-1}\}$:

$$\mathcal{M}(G,\mathcal{S}) = \mathrm{conv}\Big\{(e^{2i\pi x/N}, e^{2i\pi(N-1)x/N}) : x \in \mathbb{Z}_N\Big\}$$
$$\cong \mathrm{conv}\Big\{(\cos(2\pi x/N), \sin(2\pi x/N)) : x \in \mathbb{Z}_N\Big\}$$



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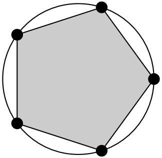

- $\mathcal{S} = \{\chi_{N-d}, \ldots, \chi_{N-1}, \chi_1, \ldots, \chi_d\}$:

$$\mathcal{M}(G,\mathcal{S}) \cong \mathrm{conv}\Big\{\left(\cos\left(\tfrac{2\pi x}{N}\right), \sin\left(\tfrac{2\pi x}{N}\right), \ldots, \cos\left(\tfrac{2\pi dx}{N}\right), \sin\left(\tfrac{2\pi dx}{N}\right)\right) : x \in \mathbb{Z}_N\Big\} \subset \mathbb{R}^{2d}$$

trigonometric cyclic polytope

Lifts of moment polytopes

$$\mathcal{M}(G,\mathcal{S}) = \text{conv}\Big\{(\chi(x))_{\chi\in\mathcal{S}} : x \in G\Big\} \subset \mathbb{C}^{\mathcal{S}}$$

Linear functions on $\mathbb{C}^{\mathcal{S}}$ have the form:

$$y \in \mathbb{C}^{\mathcal{S}} \mapsto a_0 + \sum_{\chi\in\mathcal{S}} a_\chi y_\chi$$

Linear function is nonnegative on $\mathcal{M}(G,\mathcal{S})$ if

$$a_0 + \sum_{\chi\in\mathcal{S}} a_\chi \chi(x) \geq 0 \quad \forall x \in G.$$

Thus:

Nonnegative linear functions on $\mathcal{M}(G,\mathcal{S})$

$\leftrightarrow$

Nonnegative functions on G that are supported on $\mathcal{S}$

Sparse sum-of-squares on finite abelian group

Given G finite abelian group and $\mathcal{S} \subseteq \widehat{G}$

Find $\mathcal{T} \subseteq \widehat{G}$ such that:

any nonnegative function $f : G \to \mathbb{R}_+$ supported on $\mathcal{S}$ has a sum-of-squares certificate supported on $\mathcal{T}$, i.e.,

$$f(x) = \sum_j |f_j(x)|^2$$

where support$(f_j) \subseteq \mathcal{T}$.

$\to$ Consequence: get a psd lift of $\mathcal{M}(G, \mathcal{S})$ of size $|\mathcal{T}|$.

Flavor of result

Main result: describes combinatorial way to take
G and $\mathcal{S}$ and construct (many different) $\mathcal{T}$ s.t.

$$\begin{aligned} & f : G \to \mathbb{R} \text{ non-negative and sparse w.r.t. } \mathcal{S} \\ \Longrightarrow \quad & f \text{ is a sum of squares of functions sparse w.r.t. } \mathcal{T} \end{aligned}$$

Aim: minimize $|\mathcal{T}|$ w.r.t. choices in construction

Quadratic polynomials on $\{-1, 1\}^n$

Conjecture (Laurent 2003): If

$$f(x) = a_0 + \sum_{i<j} a_{ij} x_i x_j \quad \text{non-negative } \forall x \in \{-1, 1\}^n$$

then f is a sum of squares of polynomials of degree at most $\lceil n/2 \rceil$.

- Laurent (2003): degree at least $\lceil n/2 \rceil$ necessary
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In our language:

- Group: $G = \{-1,1\}^n \cong \mathbb{Z}_2^n$
- Characters: $\chi_S(x) = \prod_{i \in S} x_i$ (square-free polynomials)
- non-negative functions with support $\mathcal{S} = \{S : |S| \in \{0,2\}\}$

Good choices in main result $\rightarrow$ prove Laurent's conjecture

Degree d polynomials on $\mathbb{Z}_N$

$$TC_{N,2d} = \text{conv}\Big\{\left(\cos\left(\tfrac{2\pi x}{N}\right), \sin\left(\tfrac{2\pi x}{N}\right), \ldots, \cos\left(\tfrac{2\pi dx}{N}\right), \sin\left(\tfrac{2\pi dx}{N}\right)\right) : x \in \{0, 1, 2 \ldots, N-1\}\Big\} \subset \mathbb{R}^{2d}$$

Trigonometric cyclic polytope

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Good choices in main result + duality:
If d divides N then $TC_{N,2d}$ has a PSD lift of size $\leq 3d \log_2(N/d)$.

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$TC_{d^2,d}$ polytope in $\mathbb{R}^{2d}$ with d^2 vertices:

- SDP lift of size $O(d \log(d))$
- LP lift must have size $\geq \Omega(d^2)$
  (lower bound due to Fiorini et al. for d-neighborly polytopes)

Main result

Assume: $f : G \to \mathbb{R}$ non-negative, sparse w.r.t. $\mathcal{S}$

Let: $\mathrm{Cay}(\widehat{G}, \mathcal{S})$ be the Cayley graph of $\widehat{G}$ w.r.t. $\mathcal{S}$

Choose:

- a chordal cover Γ of $\mathrm{Cay}(\widehat{G}, \mathcal{S})$
- and for each maximal clique C of Γ choose:
  - a character $\chi_C \in \widehat{G}$

Then: f is a sum of squares of functions supported on

$$\mathcal{T} = \bigcup_C (\chi_C \cdot C)$$

Aim: make choice of Γ and the $(\chi_C)_C$ so $\mathcal{T}$ as small as possible!

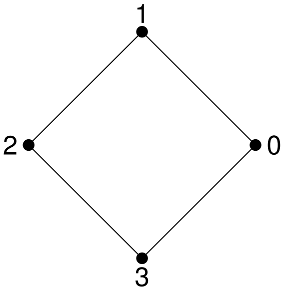
Example:

Assume: $f : \mathbb{Z}_4 \to \mathbb{R}$ is non-negative and

$$f(x) = a_0\chi_0(x) + a_1\chi_1(x) + \overline{a_1}\chi_{-1}(x)$$

i.e. sparse w.r.t. $\mathcal{S} = \{-1, 0, 1\}$.

- $\mathrm{Cay}(\widehat{\mathbb{Z}}_4, \mathcal{S})$ is the 4-cycle
- Choose: Γ a chordal cover maximal cliques $C_1 = \{0, 1, 2\}, C_2 = \{0, 2, 3\}$
- Choose: character for each maximal clique



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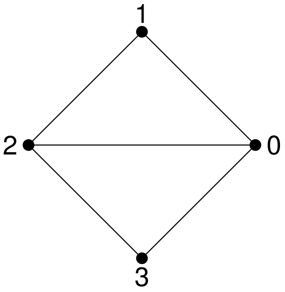

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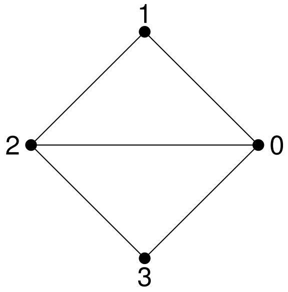

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$$\mathcal{T} = (\chi_{-1} \cdot \{0, 1, 2\}) \cup (\chi_1 \cdot \{0, 2, 3\}) = \{-1, 0, 1\} \quad \to \quad \text{size } 3$$

Proof of main theorem

Three steps:

1. A “distinguished” sum-of-squares representation with a sparse Gram matrix
2. Chordal cover and decomposition of Gram matrix
3. Translation of frequencies

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1. **A "distinguished" sum-of-squares representation with a sparse Gram matrix**
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A distinguished sum-of-squares representation

SOS in Gram matrix form:

$$f \text{ sum-of-squares } \Leftrightarrow \exists Q \succeq 0 \text{ s.t. } f(x) = [\chi(x)]^* Q[\chi(x)].$$

Q called a *Gram matrix*.
Proof of $\Leftarrow$: If $Q = \sum_k v_k v_k^* \succeq 0$ then $f(x) = \sum_k |v_k^*[\chi(x)]|^2$.

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A distinguished representation Let $f : G \to \mathbb{R}_+$ and define Q by:

$$Q_{\chi,\chi'} = \widehat{f}(\overline{\chi}\chi') \quad \forall \chi, \chi' \in \widehat{G}.$$

Then Q is positive semidefinite (eigenvalues are $\{f(x) : x \in G\}$) and

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Remark: If supp $f = \mathcal{S}$, then Q has the sparsity pattern of $\text{Cay}(\widehat{G}, \mathcal{S})$:

$$Q_{\chi,\chi'} \neq 0 \Leftrightarrow \overline{\chi}\chi' \in \mathcal{S} \Leftrightarrow \{\chi, \chi'\} \in \text{Cay}(\widehat{G}, \mathcal{S})$$

Proof of main theorem

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2. **Chordal cover and decomposition of Gram matrix**
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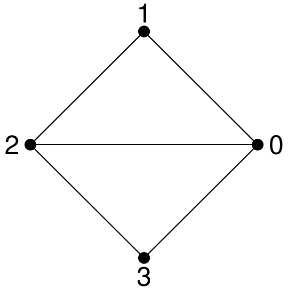
Positive semidefinite matrices with chordal sparsity

Theorem (Griewank et al., Grone et al., 1984)

If $Q \succeq 0$ and sparse w.r.t. chordal graph Γ, then Q decomposes as sum of psd matrices each supported on a maximal clique of Γ.



Gram matrix decomposes as sum of psd matrices:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & 0 \\ * & * & * & * \\ * & 0 & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} * & 0 & * & * \\ 0 & 0 & 0 & 0 \\ * & 0 & * & * \\ * & 0 & * & * \end{bmatrix}$$

Positive semidefinite matrices with chordal sparsity

$f : G \to \mathbb{R}_+$ and $\operatorname{supp} f \subseteq \mathcal{S}$. We saw

$$f(x) = \frac{1}{|G|}[\chi(x)]^* Q[\chi(x)]$$

where $Q \succeq 0$ and sparse w.r.t. $\operatorname{Cay}(\widehat{G}, \mathcal{S})$.

Proposition

Let Γ be a chordal cover for $\operatorname{Cay}(\widehat{G}, \mathcal{S})$. If f is a nonnegative function supported on $\mathcal{S}$ then it has a decomposition

$$f(x) = \sum_j |f_j(x)|^2$$

where each f_j is supported on a maximal clique of Γ.

Proof of main theorem

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Translation of frequencies

We have:

$$f(x) = \sum_j |f_j(x)|^2 \quad \text{with} \quad \operatorname{supp} f_j \subseteq C_j \quad \text{(maximal clique of } \Gamma\text{)}.$$

Problem: each C_j is maybe small, but the union of C_j's is big (it is all of $\widehat{G}$).

Main idea: We can "translate" the supports of f_j. Let $\chi_j \in \widehat{G}$. Then:

- $|\chi_j f_j|^2 = |f_j|^2$
- $\operatorname{supp}(\chi_j f_j) \subseteq \chi_j C_j$.

Thus if we let $\tilde{f}_j = \chi_j f_j(x)$ we get:

$$f(x) = \sum_j |\tilde{f}_j(x)|^2 \quad \text{with} \quad \operatorname{supp} \tilde{f}_j \subseteq \chi_j C_j$$

Main result (again)

Assume: $f : G \to \mathbb{R}$ non-negative, sparse w.r.t. $\mathcal{S}$

Let: $\mathrm{Cay}(\widehat{G}, \mathcal{S})$ be the Cayley graph of $\widehat{G}$ w.r.t. $\mathcal{S}$

Choose:

- a chordal cover Γ of $\mathrm{Cay}(\widehat{G}, \mathcal{S})$
- and for each maximal clique C of Γ choose:
  - a character $\chi_C \in \widehat{G}$

Then: f is a sum of squares of functions supported on

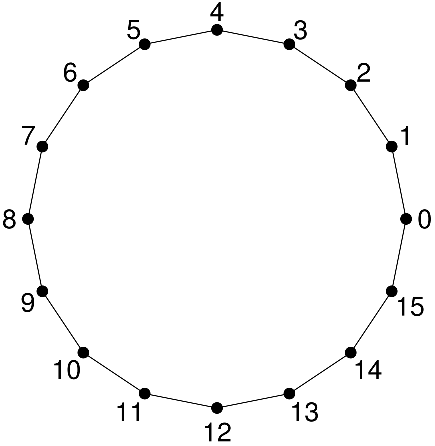
$$\mathcal{T} = \bigcup_C (\chi_C \cdot C)$$

Aim: make choice of Γ and the $(\chi_C)_C$ so $\mathcal{T}$ as small as possible!

Degree d polynomials on $\mathbb{Z}_N$

$d = 1$

- $\mathcal{S} = \{-1, 0, 1\}$
- $\text{Cay}(\widehat{\mathbb{Z}}_N, \mathcal{S})$ is N-cycle



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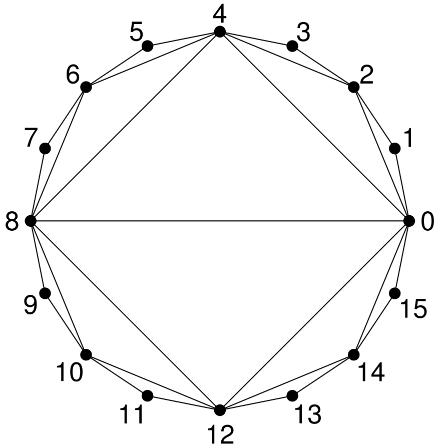

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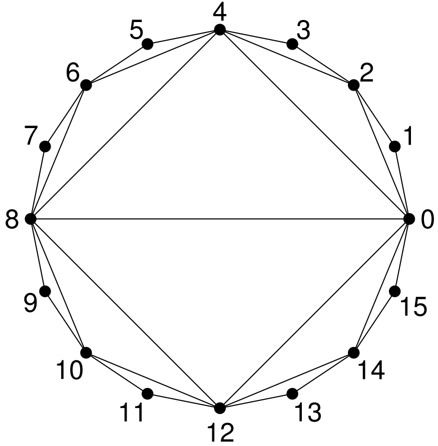
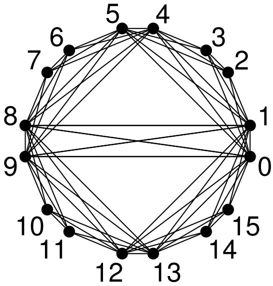
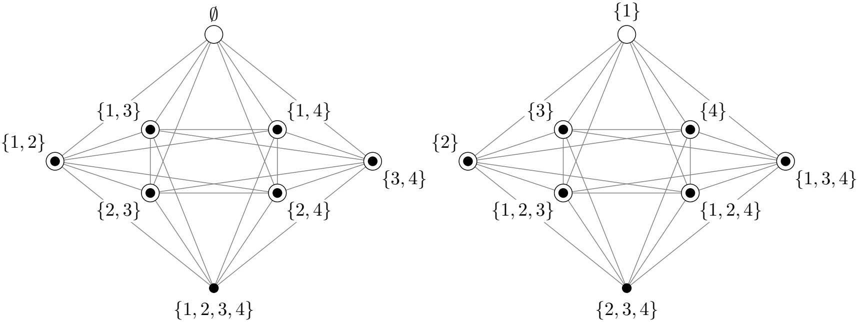




General d

- builds on $d = 1$ case
- more complicated

Quadratic functions on $\{-1, 1\}^n$



- $G = \mathbb{Z}_2^n$, $\mathcal{S} = \{S : |S| \in \{0, 2\}\}$
- $\mathrm{Cay}(\widehat{\mathbb{Z}_2^n}, \mathcal{S})$ is the half-cube graph
- the most obvious chordal cover *almost* works

Conclusion

Summary:

- If $f : G \to \mathbb{R}$ is non-negative and sparse w.r.t. $\mathcal{S}$ described way to construct $\mathcal{T}$ s.t. f is SOS of functions sparse w.r.t. $\mathcal{T}$.
- Applied to non-negative quadratics on $\{-1, 1\}^n$
  - all are SOS of functions of degree $\leq \lceil n/2 \rceil$
- Applied to non-negative degree d polynomials on $\mathbb{Z}_N$
  - explicit family of polytopes with separation between SDP and LP lits.

Questions:

- Lower bounds?
- Other interesting choices of group G and support $\mathcal{S}$?

For more information: preprint `arXiv:1503.01207`

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Thank you!