Prograde and Retrograde Motion in a Fluid Layer: Consequences for Thermal Diffusion in the Venus Atmosphere

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Depending on the value of the Prandtl number, the average velocity imparted to a layer of Boussinesq fluid by traveling thermal waves applied at the upper free surface is found, in the linear case, to be either in the same or opposite direction as that of the moving thermal source. Since the mean flow is in the opposite direction only when the Prandtl number is small, the 4-day retrograde zonal motion of the Venus atmosphere may be evidence that the effective Prandtl number of the upper atmosphere is much less than unity.

It has been proposed that the observed 4-day retrograde circulation of the Venus atmosphere is a zonal motion of at least the upper atmosphere driven by periodic solar thermal forcing [*Schubert and Whitehead,* 1969; *Schubert and Young,* 1970]. If this mechanism is indeed operative in the Venus atmosphere, the analysis of the motions induced in a fluid layer by moving thermal sources can provide information on the diffusive properties of the atmosphere.

In this paper we consider the flow induced by traveling thermal waves in a layer of Boussinesq fluid bounded above and below by horizontal free and rigid surfaces, respectively. The thermal boundary conditions will be of two types: prescribed temperature fluctuations at the free surface and fixed temperature at the rigid surface; prescribed fluctuations in heat flux at the free surface and a thermally insulating rigid surface. The heat flux boundary conditions have previously been considered by *Malkus* [1970].

The major result of our investigation is that the average motion of the fluid is either prograde (in the direction of the moving thermal wave) or retrograde depending on the magnitude of the Prandtl number $P = \nu/\kappa$ (ν is the kinematic viscosity and κ is the thermal diffusivity). The value of Prandtl number P_c at which the mean velocity at the upper free surface changes from prograde to retrograde is a function of the frequency parameter $S = \omega h^2/\nu$ (ω is the circular frequency of the thermal forcing and h is the depth of the fluid layer). For $P < P_c(S)$ the mean velocity at the upper free surface is retrograde, whereas for $P > P_c(S)$ it is prograde. In the limit $S \ll 1$, $P_c \approx 0.2$. As S becomes large, the change in character of the flow occurs at smaller values of the Prandtl number, $P_c(S) \propto S^{-3/2}$ for $S \gg 1$; at $S = 100$ (an estimate for the Venus atmosphere, *Schubert and Young* [1970]) $P_c \approx 0.025$ for heat flux boundary conditions. These results indicate that the effective Prandtl number for at least the upper regions of the Venus atmosphere is much less than unity.

ANALYSIS OF INDUCED MEAN MOTION

Consider the two-dimensional motion of a layer of Boussinesq fluid defined by horizontal planes separated by a distance h. The linearized equations of motion and energy have been considered in a number of related investigations [*Stern,* 1959; *Davey,* 1967; *Schubert,* 1969; *Kelly and Vreeman,* 1970; *Malkus,* 1970; *Whitehead,* 1970; *Hinch and Schubert,* 1971]. The following are the pertinent equations in this discussion:

$$\left(\frac{\partial^2}{\partial z^2} - iS\right)\frac{\partial^2 \tilde{w}}{\partial z^2} = \beta SF\tilde{T} \qquad (1)$$

$$\frac{\partial^2 \tilde{T}}{\partial z^2} = iSP\tilde{T} \qquad (2)$$

$$\frac{d\langle u\rangle}{dz} = \frac{S}{2\beta^2}\ \text{Imaginary}\left(\tilde{w}\,\frac{d\tilde{w}^*}{dz}\right) \qquad (3)$$

where z is the dimensionless vertical coordinate ($z = 0$ is the lower rigid surface and $z = 1$ is the upper free surface), β is the quantity kh (k is the wave number of the thermal forcing) assumed to be much less than unity, $\langle u\rangle$ is the dimensionless horizontal velocity averaged over a wavelength, and the asterisk denotes the complex conjugate. The fluctuations in the dimensionless temperature and vertical velocity are given by

$$\text{Real}\,[(\tilde{T}, \tilde{w})\ \exp\{i(kx + \omega t)\}]$$

respectively, where x is the horizontal coordinate. The velocities are dimensionless with respect to $U = \omega/k$, the speed of the thermal wave traveling in the negative x direction. The complex amplitude of the temperature fluctuation is dimensionless with respect to $\tilde{T}_w$, the real amplitude of the given temperature fluctuation at the free surface when temperature boundary conditions are applied. When heat flux boundary conditions are used, $\tilde{T}$ is dimensionless with respect to $\tilde{q}h/K$, where $\tilde{q}$ is the real amplitude of the given periodic heat flux at the upper free surface and K is the thermal conductivity. Finally the parameter F equals $gh\alpha\tilde{T}_w/U^2$, for temperature boundary conditions, or (gh/U^2) $(\tilde{q}\alpha h/K)$ for heat flux boundary conditions, where α is the coefficient of thermal expansion and g is the acceleration of gravity. Equations 1–3 are solved for each of the following sets of boundary conditions:

$$\tilde{w} = \frac{d\tilde{w}}{dz} = \langle u\rangle = \tilde{T} = 0 \quad \text{at } z = 0$$
$$\tilde{w} = \frac{d^2\tilde{w}}{dz^2} = 0,\ \tilde{T} = 1 \quad \text{at } z = 1 \qquad (4)$$

and

$$\tilde{w} = \frac{d\tilde{w}}{dz} = \langle u\rangle = \frac{d\tilde{T}}{dz} = 0 \quad \text{at } z = 0$$
$$\tilde{w} = \frac{d^2\tilde{w}}{dz^2} = 0,\ \frac{d\tilde{T}}{dz} = 1 \quad \text{at } z = 1 \qquad (5)$$

The general solutions for $\tilde{w}$ and $\tilde{T}$ are easily obtainable but algebraically complicated. Since our main interest here is in the mean motion $\langle u\rangle$, we present the results for $\langle u\rangle$ at the upper free surface ($\langle u\rangle(1)$) and for the vertically averaged mean velocity $\langle u_{avg}\rangle$ in the limiting cases $S, SP \ll 1$, $S, SP \gg 1$, and $S^{-1/2}, SP \ll 1$. For the mean velocity at the upper free surface and for the vertically averaged mean velocity, we find, in the limit $S, SP \ll 1$

$$\langle u\rangle(1) \approx \frac{F^2 S^4(93 - 544P)}{(5!)^2(4!)(7!)7} \qquad (6)$$

$$\langle u_{avg}\rangle \approx \frac{F^2 S^4}{14!}\left(\frac{947}{2} - \frac{8143}{5}P\right) \qquad (7)$$

with temperature boundary conditions and

$$\langle u\rangle(1) \approx \frac{F^2 S^4(9 - 56P)}{14(5!)^2(4!)^2(S^2P^2 + \beta^4)} \qquad (8)$$

$$\langle u_{avg}\rangle \approx \frac{3F^2 S^4(9 - 46P)}{16(11!)(S^2P^2 + \beta^4)} \qquad (9)$$

with heat flux boundary conditions. In obtaining equations 8 and 9 it is important to replace $\partial^2/\partial z^2$ in equations 1 and 2 by $\partial^2/\partial z^2 - \beta^2$; $\langle u\rangle$ (1) and $\langle u_{avg}\rangle$ for the heat flux boundary conditions would otherwise behave in a singular manner as $SP \rightarrow 0$.

In the low-frequency limit $S, SP \ll 1$ both the mean velocity at the upper free surface and the vertically averaged mean velocity can be either prograde or retrograde depending on the value of the Prandtl number. For the temperature boundary conditions, $\langle u\rangle(1)$ is prograde when $P \gtrsim 0.171$ ($P_c = 0.171$) and $\langle u_{avg}\rangle$ is prograde when $P \gtrsim 0.291$. Thus there is a range of Prandtl numbers, from approximately 0.171 to about 0.291, when the vertically averaged mean flow is retrograde while the velocity at the upper free surface is prograde. For the heat flux boundary conditions $\langle u\rangle(1)$ and $\langle u_{avg}\rangle$ change from prograde to retrograde at $P \approx 0.161$ and $P \approx 0.196$, respectively. Thus the qualitative nature of these results appears not to be effected by the particular thermal boundary conditions.

In the high-frequency limit $S, SP \gg 1$ we find

$$\langle u\rangle(1) \approx \frac{F^2(P^2+1)}{4S(P-1)^2P^2} \cdot\left\{\frac{4P^{3/2}}{(1+P^2)(1+P)} - 1\right\} \quad (10)$$

$$\langle u_{avg}\rangle \approx \frac{F^2}{4(2)^{1/2}S^{3/2}(P-1)(P+1)^2P^{5/2}} \cdot\{(7P+3) - P^{1/2}(2P^3+6P^2+P+1)\} \quad (11)$$

for the temperature boundary conditions, while for the heat flux boundary conditions $\langle u\rangle(1)$ and $\langle u_{avg}\rangle$ are given by the products of $1/SP$ with equations 10 and 11, respectively (with F properly interpreted).

Both $\langle u\rangle(1)$ and $\langle u_{avg}\rangle$ are always prograde in the limit S, $SP \gg 1$. However, for P sufficiently small, when SP is not large, the flow will be retrograde. This may be seen by investigating the limit $S^{-1/2}$, $SP \ll 1$, wherein

$$\langle u\rangle(1) \approx \frac{F^2S}{6!\,4!\,(2)^{1/2}}\left(\frac{120}{S^{1/2}} - (2)^{1/2}SP\right) \quad (12)$$

$$\langle u_{avg}\rangle \approx \frac{F^2S(2)^{1/2}}{7!\,6!}\left(\frac{8820}{S^{1/2}} - 41SP\right) \quad (13)$$

for temperature boundary conditions and

$$\langle u\rangle(1) \approx \frac{F^2S}{2(2)^{1/2}6!\,(S^2P^2+\beta^4)} \cdot\left(\frac{60}{S^{1/2}} - (2)^{1/2}SP\right) \quad (14)$$

TABLE 1. The Values of Prandtl Number at Which $\langle u\rangle(1)$ Changes from Prograde to Retrograde for Heat Flux Boundary Conditions

S	P_c (numerical)	$P_c(S \ll 1)$	$P_c(S^{-1/2}, SP \ll 1)$
0.1	0.161	0.161	
1	0.161	0.161	
10	0.147	0.161	
20	0.119	0.161	0.474
30	0.0924	0.161	0.258
50	0.0576	0.161	0.120
100	0.0252	0.161	0.0424
200	0.0103	0.161	0.015
500	0.00298	0.161	0.00379
600	0.00225	0.161	0.00289
700	0.00185	0.161	0.00229
800	0.00167	0.161	0.00188

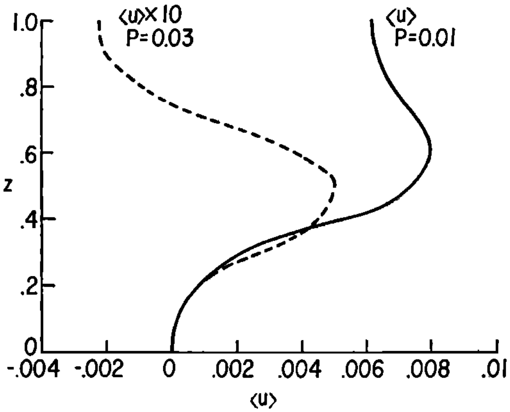

Fig. 1. Average velocity profiles with heat flux boundary conditions for $P = 0.01$ and $P = 0.03$ at $S = 100$, $F = 0.3$.

$$\langle u_{avg}\rangle \approx \frac{F^2S}{(\sqrt{2}\,8!\,(S^2P^2+\beta^4)} \cdot\left(\frac{1260}{S^{1/2}} - 13(2)^{1/2}SP\right) \quad (15)$$

for heat flux boundary conditions (the remarks which follow equations 8 and 9 are pertinent to the results obtained in equations 14 and 15). The dependence of P_c on S is seen from equation 12 to be $P_c \approx 84.85\ S^{-3/2}$ for temperature boundary conditions and from equation 14 to be $P_c \approx 42.43\ S^{-3/2}$ for heat flux boundary conditions. The vertically averaged mean velocity changes from prograde to retrograde at $P \approx 215.12\ S^{-3/2}$ for temperature boundary conditions and $P \approx 68.53\ S^{-3/2}$ for heat flux boundary conditions. Thus we see again that the mass flow may be retrograde while the mean velocity at the upper free surface is prograde.

We have numerically computed the solution of equations 1–3 subject to the conditions given in (5) and have determined $P_c(S)$. These values of P, below which $\langle u\rangle(1)$ is retrograde are given in Table 1 as a function of S. The critical values of P, computed from the asymptotic formulas 8 and 14 are also listed in the table. We note in particular that for $S = 100$ (an estimate for the Venus atmosphere) $\langle u\rangle(1)$ is prograde or retrograde according to whether $P \gtrsim$ or $\lesssim 0.025$, respectively. Two mean velocity profiles, illustrating the nature of the flows when the motion at the upper surface is either prograde or retrograde, are shown in Figure 1 for the case of heat flux boundary conditions with $S = 100$ and $F = 0.3$. In the linear problem we note that $\langle u \rangle$ scales with F^2. The formulas given by *Malkus* [1970] for $\langle u \rangle(1)$ in the limits of small and large frequency for the heat flux boundary conditions (equations (2.20) and (2.21) in that paper) are incorrect, apparently because of algebraic errors incurred in the process of finding the limiting forms of the general solution.

The physical mechanism responsible for these prograde and retrograde flows is the tilting of the traveling convection cells. As can be seen from the preceding results, the downward diffusion of the thermal field produces a tilt of these cells in the direction of the traveling thermal wave that corresponds to a mean upward transport of prograde momentum. Viscous diffusion from the lower rigid surface yields a tilt of the convection cells in the oppsite sense corresponding to a mean upward transport of retrograde momentum. The role of thermal diffusion is to produce prograde mean flow, whereas that of viscous diffusion leads to retrograde mean flow. Only where heat is well diffused ($P \ll 1$) and where there is relatively little thermal tilting of the convection cells can retrograde flow occur.

HEAT AND MOMENTUM TRANSPORT IN THE VENUS ATMOSPHERE

The results of the preceding discussion show that for a given frequency parameter either prograde or retrograde mean flow is possible, depending on the value of the Prandtl number. In obtaining these results we have considered only the linear problem (we have assumed constant thermal and momentum-diffusion coefficients) and used the Boussinesq approximation. Further, we have chosen simple thermal and velocity boundary conditions. However these boundary conditions should be physically relevant to a planetary atmosphere. In this connection we also note that the two sets of boundary conditions used in the analysis led to qualitatively similar results. With these remarks in mind, we conclude that the observed 4-day retrograde zonal motion of the Venus atmosphere indicates that the effective Prandtl number of at least the upper regions of the atmosphere is small compared with unity.

If it is assumed that momentum diffusion is described by an effective eddy viscosity, a small Prandtl number indicates that thermal diffusion is caused by some process other than turbulence. In the upper atmosphere of Venus heat transport is thus likely to be radiative. *Avduevsky et al.* [1970*a*] have concluded that the thermal balance above altitudes of 40–50 km is radiative, whereas at lower altitudes heat transport must be convective. Let us then assume that the upward traveling long-wave radiative flux required to balance the incoming solar radiation can be described in terms of an effective thermal conductivity and a temperature gradient [*Goody and Robinson*, 1966]. On the basis of the data from Veneras 4, 5, and 6, *Avduevsky et al.* [1970*a*, *b*] give a value of the temperature gradient approximately equal to 7×10^{-5} °K/cm, densities that range between 2×10^{-3} and 1.2×10^{-4} g/cm^3 between altitudes of 50 and 70 km, and a value of the upward radiative flux above 50 km of about 150 watts/m^2. These values yield a radiative thermal diffusivity that ranges between 10^5 to 2×10^6 cm^2/sec. If we use a value for the eddy viscosity of 10^4 cm^2/sec [*Goody and Robinson*, 1966], we get a Prandtl number ranging from 0.1 to 0.005, which is in the range that gives retrograde mean flow for $S \gtrsim 100$.

Acknowledgment. Gerald Schubert and Richard E. Young acknowledge support under NSF grant GA 10167.

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(Received October 8, 1970;
revised December 21, 1970.)