Viscoelastic flow through a contraction

Progress and Problems

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Motivation

Refereeing :

Boyko, E. & Stone, H. A. (2022) *Pressure-driven flow of the viscoelastic Oldroyd-B fluid in narrow non-uniform geometries: analytical results and comparison with simulations.* JFM 936

Recall:

Hinch, E.J. (1993) *The flow of an Oldroyd fluid around a sharp corner.* JNNFM 40

Realise: A new application for an old FAST trick!

Simplifications

1. Slowly varying = lubrication theory, with boost for tension in streamlines
2. Oldroyd-B

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu_0\mathbf{e} + G\mathbf{A}.$$

Microstructure **A** deforms with the flow (upper-convective time derivative), and relaxes

$$\overset{\triangledown}{\mathbf{A}} \quad = \quad \frac{D\mathbf{A}}{Dt} - \mathbf{A} \cdot \boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\mathbf{u}^T \cdot \mathbf{A} \quad = \quad -\frac{1}{\tau}\left(\mathbf{A} - \mathbf{I}\right).$$

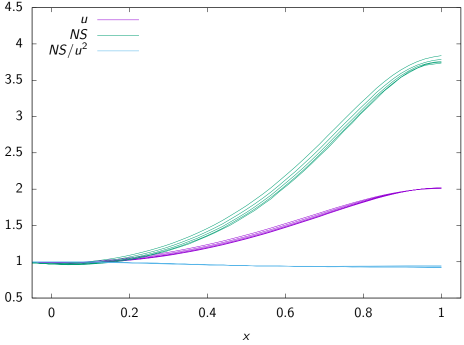
Fast flow trick

Fast $=$ no time to relax

$$\overset{\triangledown}{\mathbf{A}} = \frac{D\mathbf{A}}{Dt} - \mathbf{A} \cdot \nabla\mathbf{u} - \nabla\mathbf{u}^T \cdot \mathbf{A} = -\frac{1}{\tau}(\mathbf{A} - \mathbf{I}).$$

Solution: $\mathbf{A} = \lambda(\psi)\mathbf{uu}$

Plot variation along streamlines through the contraction

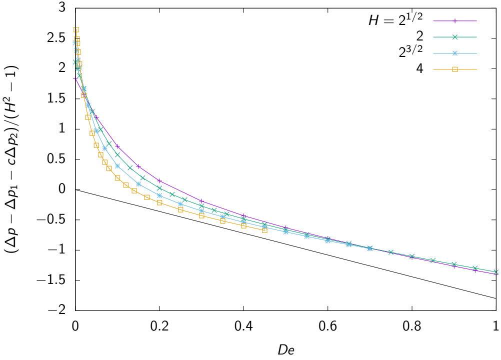


Conclude $A \propto u^2$ on streamlines

Fast flow 2

Hence predict stress and pressure drop for large De (> 0.3)

$$\Delta p = \Delta p_1 + c\Delta p_2 - \frac{9cDe}{5}\left(H^2 - 1\right).$$



Different planar contraction ratios, $c = 1.0$.

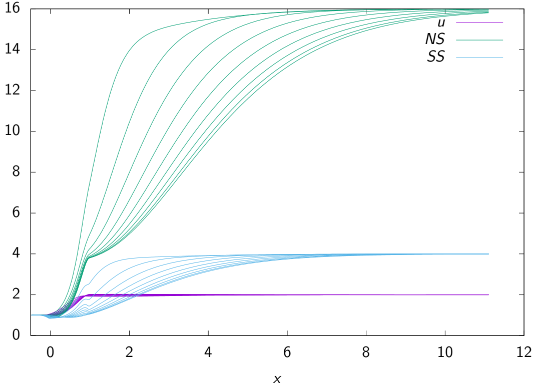
Problem 1. Relaxation after the contraction

Too many recent papers have just found the pressure drop in the contraction .

Contraction in $0 \leq x \leq 1$

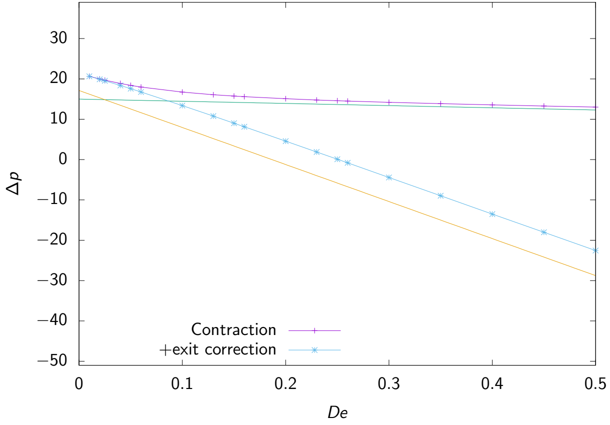
Planar contraction
$De = 0.5,\ H = 2,\ c = 1.0$

Need to go to $x = 8$
for 95% relaxation



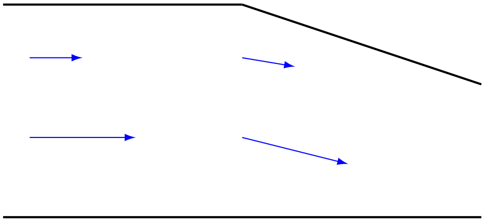
exit channel correction

Planar contraction $De = 0.5$, $H = 2$, $c = 1.0$



Neglected exit channel dominates change in pressure drop!

Problem 2. Non-smooth boundary



Discontinuity of slope of boundary,
$\rightarrow$ discontinuity in y-velocity
(in lubrication theory picture),
$\rightarrow$ δ-function of vorticity,
$\rightarrow$ jump in direction of elastic stresses

$$\begin{aligned}
A_{xx}(0+) &= A_{xx}(0-), \\
A_{xy}(0+) &= A_{xy}(0-) + \eta A_{xx}(0-), \\
A_{yy}(0+) &= A_{yy}(0-) + 2\eta A_{xy}(0-) + \eta^2 A_{xx}.
\end{aligned}$$

where $\eta = yH'(0+)/H^2(0)$

Avoid with smooth boundary or streamline coordinates

Problem 3. Naive small-*De* expansion

$$\overset{\triangledown}{\mathbf{A}} \quad = \frac{D\mathbf{A}}{Dt} - \mathbf{A} \cdot \boldsymbol{\nabla}\mathbf{u} - \boldsymbol{\nabla}\mathbf{u}^T \cdot \mathbf{A} = -\frac{1}{De}\left(\mathbf{A} - \mathbf{I}\right).$$

Iterate

$$\mathbf{A} = \mathbf{I} \; - \; De \overset{\triangledown}{\mathbf{I}} \; + De^2 \; \overset{\triangledown\triangledown}{\mathbf{I}} \; - De^3 \; \overset{\triangledown\triangledown\triangledown}{\mathbf{I}} \; + De^4 \; \overset{\triangledown\triangledown\triangledown\triangledown}{\mathbf{I}} + \dots$$

with no opportunity to satisfy inlet stress boundary condition.

Several recent papers have made this naive expansion to many, $O(De^8)$, erroneous terms.

For hyperbolic contraction

$$R(z) = \frac{1}{(1+\beta z)^{1/2}}$$

with jump in shape (so fail to satisfy inlet condition)

$$A_{22} = 1 - De\beta F + De^2\beta^2 F^2 - De^3\beta^3 F^3 + \cdots .$$

Summing

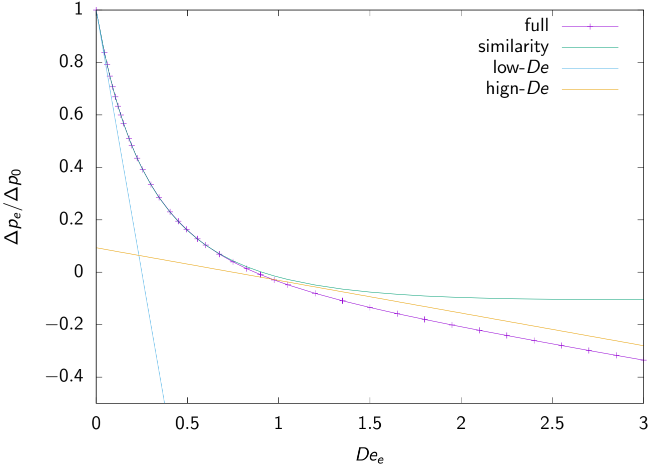
$$A_{22} = \frac{1}{1 + De\beta F}.$$

and similar for A_{12} and A_{11}.

This is the exact/similarity solution of Sialmas & Housiadas, NOT satisfy inlet stress condtion.

Panagiotis Sialmas & Kostas D. Housiadas (2025) *An exact solution of the lubrication equations for the Oldroyd-B model in a hyperbolic pipe* JNNFM 335

Problem 4: Similarity solutions fails badly at $De_e > 1$



Integrating pressure gradient

$$\int_0^1 \frac{dz}{(1+\beta z)^{1/De-1}}$$

Dominated by x near 1
to all $O(De^n)$ for $De < 0.5$

Then switch to whole range

Conclusions

- Progress: Fast flow
- Problem: need long exit channel
- Problem: need smooth geoemtry
- Problem: small-*De* expansion could be naive (wrong)
- Problem: a new similarity solutions fails badly at $De_e > 1$