The much-neglected Second Normal Stress Difference

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What are normal stresses?

In simple shear $\mathbf{u} = (\gamma y, 0, 0)$, the stress tensor is

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix},$$

with off-diagonal tangential viscous dissipative stresses, and diagonal normal stresses, which do no work.

In a Newtonian viscous fluid, the normal stresses are equal $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$ and equal to (negative) pressure.

In a visco-elastic fluid, the normal stresses are not equal. We consider their differences

$$N_1 = \sigma_{xx} - \sigma_{yy}, \quad N_2 = \sigma_{yy} - \sigma_{zz}.$$  

$N_1$ (dominant for polymers) is a tension in the streamlines.
Effect of $N_1$ on flow – tension in the streamlines

- Rod climbing
- Secondary flows
- Migration of particles to centre of pipe flow
- Stabilisation of jets
- Purely-elastic Taylor-Couette instability
- Co-extrusion instability
The origin of $N_1$

For polymers and microstructures of fibres:

Fibres in shear flow become stretched and aligned

Thus tension in the streamlines, $N_1$.

And no $N_2$ from fibres – why much neglected.
The origin of $N_2$

Need a thick microstructure that can be compressed, e.g. droplets in an emulsion.

Then spin by vorticity to align with flow gives $N_2 = \sigma_{yy} - \sigma_{zz} < 0$.

$N_2 < 0$ is tension in the vortex lines.
Thick microstructures, unlike thin fibres, strain with reduced efficiency, $\theta < 100\%$.

So deform with

$$\nabla u \rightarrow \frac{1}{2} \left( \nabla u - \nabla u^T \right) + \theta \frac{1}{2} \left( \nabla u + \nabla u^T \right)$$

This gives a second normal stress difference and shear-thinning.

$$N_2 \propto -\frac{\theta(1 - \theta)\gamma^2}{1 + (1 - \theta^2)\gamma^2}, \quad \mu_p \propto \frac{\theta}{1 + (1 - \theta^2)\gamma^2},$$
Another origin of $N_2$

In non-Brownian suspensions, particles impact in the $x$-direction, leading to a pressure, $\sigma_{xx} < 0$.

When concentrated $\phi > 20\%$, they also impact layer above and below, leading to a similar pressure $\sigma_{yy} \approx \sigma_{xx}$ (force-chains at $45^\circ$ to flow), so $N_1 \approx 0$.

Easier to pass in $z$-direction, so $\sigma_{zz} \approx 0$, so $N_2 = \sigma_{yy} - \sigma_{zz} < 0$.

Boyer, Pouliquen & Guazzelli (2017)
Effect of $N_2$ on flow – tension in the vortex lines

- Bowing of interface in Tanner tilted channel
- Longitudinal vortices in granular chute flow
- Negative rod-climbing
- Edge instability in rheometers
- Lopsided de-wetting on a vertical fibre
Tanner’s tilted trough

Inclined V-shaped open channel

Higher shear-rate in centre
→ higher tension in vortex lines in centre,
→ pull fluid to centre
→ surface bows up

Same mechanism for longitudinal vortices in granular chute flow?
Negative rod-climbing

Standard analysis
Beavers & Joseph 1975

\[ h(r) = \frac{1}{\rho g} \left( \frac{1}{4} N_1 + N_2 \right) \frac{a^2}{r^2} \]

For polymers, \( N_1 > 0 \) and \( N_2 \approx 0 \), so climbs \( h > 0 \), by tension in streamlines

For concentrated non-Brownian suspensions, \( N_1 \approx 0 \) and \( N_2 < 0 \), so \( h < 0 \) dips (negative climb)
by tension in the vertical vortex lines

Boyer, Poulquen & Guazzelli 2017
Edge instability in rheometer

At edge of plate-plate rheometer, top plate coming towards you, bottom away. Perturb liquid interface, in at $A$.

Contours of constant $u$ must meet interface at $90^\circ$.

Crowding of contours at $A$,

$\rightarrow$ increase shear-rate at $A$,

$\rightarrow$ higher tension in vortex lines at $A$,

$\rightarrow$ pulls $A$ further into liquid.

Hemmingway & Fielding 2017, Tanner 1993
Lopsided de-wetting of coating on a vertical fibre

Boulogne, Pauchard & Giorgiutti-Dauphiné 2012

Thicker side

→ higher shear-rate

→ higher tension in vortex lines

→ pulls round to make thicker
Lopsided de-wetting of coating on a vertical fibre

Lubrication equations for thin coating.
Case no $z$-variations, $h(\theta, t)$

$$h_t + \left( h^3 (h_{\theta\theta} + h + B h^2) \right)_\theta = 0,$$
where $B = \frac{\psi g^2}{\delta \mu \nu^2 a^2}$.

$h(\theta, t)$ at $t = 2^n$
$n = -2, \ldots, 11,$ for $B = 0.5$.

Dotted blue is a steady state which wets only $0 \leq \theta \leq 1.9071$
Lopsided de-wetting of coating on a vertical fibre
Draining of small region to right

Small region drains as $t^{-1/4}$

$$h(\pi) \sim \frac{1 + \cos L}{t^{1/4}} \left( \frac{K((\pi - L) \cos L + \sin L)}{4Q \sin^5 L} \right)^{1/4}$$

cf P.S. Hammond 1983
Effect of $N_2$ on flow – tension in the vortex lines

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$N_2$ should not have been neglected!
And Anthony . . .

The last problem combines

- Practical application of mathematics - L’Oréal
- Non-Newtonian fluid mechanics
- Surface tension, if not Marangoni
- Asymptotic analysis

Four aspects of Anthony’s pioneering work of the 1950s.

Happy birthday Anthony