A load of balls in Newton’s cradle
or Fragmentation of a line of balls by an impact

John Hinch

DAMTP, Cambridge

March 26, 2010
\[ t = 0 \]
\[ V=1 \quad \text{others at rest} \]

\[ \begin{array}{c}
\bigcirc \quad \text{\vector{1}} \quad \bigcirc \quad \text{\vector{1}} \quad \bigcirc \quad \text{\vector{1}} \quad \bigcirc \quad \text{\vector{1}} \quad \bigcirc \quad \text{\vector{1}} \\
\end{array} \]
\( t = 0 \)

\( V=1 \) others at rest

Contacts: linear springs in compression, zero force in tension

\[ \ddot{x}_n = (x_{n+1} - x_n)_- - (x_n - x_{n-1})_- \]

No non-dimensional groups
\[ t = 0 \]

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Contacts: linear springs in compression, zero force in tension

No non-dimensional groups

Granular media?
Position of particles

$n + x_n$

If in contact: \( x_n = x_{n+1} - 2x_n + x_{n-1} \)

Continuum approximation: \( x_{tt} = x_{nn} \), with wave speed 1.

Non-continuum effect: \( \dot{x}_n(t = \infty) < 0 \)

Few forwards
Most backwards
Position of particles

\[ n + x_n \]

If in contact: \[ \ddot{x}_n = x_{n+1} - 2x_n + x_{n-1} \]

Continuum approximation: \[ x_{tt} = x_{nn} \], with wave speed 1.
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Questions

▶ How many fly off at the far end?
   At what velocities?
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▸ How many rebound?
   At what velocities?
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  At what velocities?

- How many rebound?
  At what velocities?

Simple mechanics
  Simple questions
  Answers more complicated
Rebound velocities

Chain of 25 particle, 20 rebound

\[ -\dot{x}_n(\infty) \]

If so, rebound energy \( \sum n^{-3/4} \) is finite, rebound momentum \( \sum n^{-3/4} \) is infinite.

But why \( n^{-3/4} \)? Not diffusion.

\[ n^{-3/4} \]
Rebound velocities

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\[-\dot{x}_n(\infty)\]

\(n^{-3/4}\)?

If so

Rebound energy \(\sum n^{-3/2}\) is finite
Rebound momentum \(\sum n^{-3/4}\) is infinite

But why \(n^{-3/4}\)? Not diffusion.
Key: propagation of an impulse wave

$t = 13.5, 24.5, 35.2, 45.7$

Peak velocity decreases slowly. How?

Width of pulse increases slowly. How?
Key: propagation of an impulse wave

Peak velocity decreases slowly. How?
Width of pulse increases slowly. How?
Spreading wave which conserves energy

Slowly varying amplitude and wavelength of propagation wave of constant form $f(.)$

$$x_n = a(t) f \left( \frac{n - t}{\lambda(t)} \right)$$
Spreading wave which conserves energy

Slowly varying **amplitude** and **wavelength** of **propagation wave** of constant form \( f(.) \)

\[
\chi_n = a(t) \ f \left( \frac{n - t}{\lambda(t)} \right)
\]

Then

\[
\dot{\chi}_n \sim -\frac{a}{\lambda} f'
\]

Kinetic energy (potential energy similar)

\[
\sum \dot{\chi}_n^2 \propto \frac{a^2}{\lambda^2} \lambda
\]
Spreading wave which conserves energy

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Conservation gives $\lambda \propto a^2$, and so $\dot{x}_n \propto a^{-1}$
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\sum \dot{x}_n^2 \propto \frac{a^2}{\lambda^2} \lambda
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Replot for different \( t \), \( \dot{x}_n a \) against \( (n - n_1)/a^2 \), where \( n_1 \) is last in contact, and \( a = x_{n_1} \).
Key: propagation of an impulse wave

Replot for different $t$, $\dot{x}_n a$ against $(n - n_1)/a^2$, 

$t = 13.5, 24.5, 35.2, 45.7$
Self similar impulse wave

Here \( a = x_{n_1} \) at last in contact.
Self similar impulse wave

Here $a = x_{n_1}$ at last in contact. But how to predict $a(t)$?
Scaling for spreading wave: $\lambda(t)$?

If touching

$$\ddot{x}_n = x_{n+1} - 2x_n + x_{n-1}$$

Second approximation for continuum limit

$$x_{tt} = x_{nn} + \frac{1}{12}x_{nnnn}$$

with 'numerical diffusion'.
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Transform to moving coordinate $N = n - t$

$$x_{tt} + 2x_{Nt} = \frac{1}{12}x_{NNNN}$$
Scaling for spreading wave: $\lambda(t)$?

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Second approximation for continuum limit

$$x_{tt} = x_{nn} + \frac{1}{12}x_{nnnn}$$

with 'numerical diffusion'.

Transform to moving coordinate $N = n - t$

$$x_{tt} + 2x_N = \frac{1}{12}x_{NNNN}$$

To balance last two terms, use similarity variable $N/t^{1/3}$.

Hence $\lambda \propto t^{1/3}$
Results for spreading wave

wavelength \( \lambda \propto t^{1/3} \)
displacements \( x_n = a \propto t^{1/6} \)
velocities \( \dot{x}_n = a/\lambda \propto t^{-1/6} \)
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Forward moving momentum

\[
P = \sum \dot{x}_n \propto (a/\lambda)\lambda \propto t^{1/6}
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Rate of ejecting momentum backwards in ‘rocket effect’

\[ \dot{x}_n(\infty) \Delta t = -\dot{P} \propto t^{-5/6} \]
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Rate of ejecting momentum backwards in ‘rocket effect’

\[ \dot{x}_n(\infty) \Delta t = -\dot{P} \propto t^{-5/6} \]

\( \Delta t \) time between particles rebounding = 1 (wave speed = 1)

Hence rebound velocities

\[ \dot{x}_n(\infty) \propto -t^{-5/6} = -n^{-5/6} \]
Rebound velocities

\[ \dot{x}_n / n^{-3/4} \]

\[ \dot{x}_n / n^{-5/6} \]

\[ \dot{x}_n(\infty) = -0.158 n^{-5/6} \]
Similarity solution

Try \( x(n, t) = t^{1/6} f \left( \xi = \frac{n-t}{t^{1/3}} \right) \) in \( x_{tt} = x_{nn} + \frac{1}{12} x_{nnnn} \).
Similarity solution

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So \( f''' = 8\xi f' - 4f \)
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So \( f''' = 8 \xi f' - 4f \)

Solution \( f = \int_{\xi}^{\infty} \text{Ai}^2(-2^{1/3} y) \, dy \)
The $\frac{1}{4}$ shift

Near to back of the wave $\xi = \xi_0$

$$f \sim f(\xi_0) \left( 1 - \frac{2}{3} (\xi - \xi_0)^3 + \ldots \right)$$

Correction for ejected velocities at $t^{-5/6}$

$$x_n(t) \sim t^{1/6} f(\xi) + t^{-1/2} \beta (\xi - \xi_0)$$

Ball $n$ detaches at $t_n$ where $\xi = \xi_0 + \delta$ if $x_n(t_n) = x_{n+1}(t_n)$, i.e.

$$t^{-5/6} \left[ f(\xi_0) \left( 2\delta^2 + 2\delta - \frac{2}{3} \right) - \beta \right] = 0$$

and detaches with known velocity

$$-\frac{1}{6} f(\xi_0) t^{-5/6} = \dot{x}_n = t^{-5/6} \left[ f(\xi_0) 2\delta^2 - \beta \right]$$

Hence $\delta = \frac{1}{4}$. 
Finite chain of \( N \): number fly off and their velocities

When wave reaches end at \( t = N \).
width of wave \( 1.5N^{1/3} \) and velocities \( 1.4N^{-1/6} \) and less.
Finite chain of $N$: number fly off and their velocities

When wave reaches end at $t = N$.
width of wave $1.5N^{1/3}$ and velocities $1.4N^{-1/6}$ and less.

$N = 100 \diamond, 200 +, 400 \square, 800 \times$

$V_n N^{1/6}$

$(N - n)N^{-1/3}$
Questions: Answers

- How many fly off at the far end?
Questions: Answers

- How many fly off at the far end? $1.5N^{1/3}$
Questions: Answers

- How many fly off at the far end? \(1.5N^{1/3}\)
  
  At what velocities? \(V_N = 1.4N^{-1/6}\)
Questions:  Answers

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  At what velocities?  \(V_N = 1.4N^{-1/6}\)

- How many rebound?  Most
  At what velocities?  \(V_n = -0.16n^{-5/6}\)
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Simple mechanics
  Simple questions
    Answers more complicated
Questions: Answers

- How many fly off at the far end? 1.5N^{1/3}
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Simple mechanics
  Simple questions
  Answers more complicated

For linear force law. Next nonlinear Hertz contacts
Hertz contacts

Radius $R$
Contact radius $a$
Overlap $\delta$

Geometry:
\[ a = \sqrt{2} R \delta \]

Strain = $\delta$
Stress = $E \delta a$
Force = $\pi a^2 E \delta a$

\[ = \sqrt{2} E \frac{3(1-\nu^2)}{2} R \frac{1}{2} \delta \frac{3}{2} \]
Hertz contacts

Radius $R$
Contact radius $a$
Overlap $\delta$

Geometry: $a = \sqrt{2R\delta}$

Strain $= \delta$
Stress $= E\delta$
Force $= \pi a^2 E\delta = \sqrt{2E^3(1-\nu^2)/R}$
Hertz contacts

Geometry: \( a = \sqrt{2R\delta} \)

Strain: \( \frac{\delta}{a} \)
Hertz contacts

Geometry: \[ a = \sqrt{2R\delta} \]

Strain: \[ \delta \frac{a}{a} \]
Stress: \[ E \frac{\delta}{a} \], elastic modulus \( E \)
Hertz contacts

Geometry: \( a = \sqrt{2R\delta} \)

Strain = \( \frac{\delta}{a} \)  
Stress = \( E \frac{\delta}{a} \), elastic modulus \( E \)

Force = \( \pi a^2 E \frac{\delta}{a} = \frac{\sqrt{2E}}{3(1 - \nu^2)} R^{1/2} \delta^{3/2} \)
Impulse wave propagating down a Hertzian chain

No spreading: nonlinearity balances ‘diffusion’
Impulse wave propagating down a Hertzian chain

No spreading: nonlinearity balances ‘diffusion’

Two fly off end: $V_N = 0.984$, $V_{N-1} = 0.149$, 
($V_{N-2} = 0.004$, $V_{N-3} = 10^{-5}$, $V_{n-4} = 3 \times 10^{-8}$)
Rebound velocities for Hertzian chain

\(-x_n(\infty)\)

\[x_n(t = \infty) = -0.084e^{-0.55n}\]
Nesterenko’s (1984) solitary wave (long wave approx)

\[ \ddot{x} = D \left[ (Dx)^{3/2} \right] \quad \text{where} \quad Du = u_{n+\frac{1}{2}} - u_{n-\frac{1}{2}} \sim \frac{\partial u}{\partial n} + \frac{1}{24} \frac{\partial^3 u}{\partial n^3} \]
Nesterenko’s (1984) solitary wave \textit{(long wave approx)}

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Travelling wave solution \( x_n(t) = f(\xi = n - V t) \)

\[
V^2 f' = f'^{3/2} + \frac{1}{24} \left( f'^{3/2} \right)'' + \frac{1}{16} f'^{1/2} f'''
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with solution

\[ V^2 = \frac{4}{5} A^{1/2}, \quad f' = A \sin^4 \sqrt{\frac{2}{5}} \xi \quad 0 \leq \xi \leq \pi \]
Nesterenko’s (1984) solitary wave (long wave approx)

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Set \( A = 1.0064 \) for energy = 0.9937

Predict \( V = 0.896 \quad \text{Max} \dot{x}_n = 0.902 \quad [x_n] = 1.875 \)

Numerical \( V = 0.841 \quad \text{Max} \dot{x}_n = 0.681 \quad [x_n] = 1.354 \)
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Wave not so long with just 4 balls
Finished?
Impact by two solitary waves

Balls fly off ends: from first wave at 1.234, 0.186, from second wave at 0.647, 0.090
Impact by two

Two solitary waves

Balls fly off end:
from first wave at 1.234, 0.186,
from second wave at 0.647, 0.090
Impact by three

Three solitary waves

\[ x_n \]

From first wave at 1.336, 0.203,
second at 0.951, 0.141,
third at 0.469, 0.054
Impact by $K$

$2K$ fly off at far end

$K = 3 \Diamond, 6 +, 9 \Box, 12 \times$

$\dot{x}_n(\infty)$

$(N - n + \frac{1}{2})/K$

Open question
Impact by $K$

$\approx \frac{1}{2}K$ rebound at velocities $> 0.01$.

$K = 10 \diamond, 20 +, 30 \square, 40 \times, 60 \triangle, 80 \ast$

Open question
Newton’s cradle

Final velocities of position $n$ after impact by $K$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$K = 1$</th>
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<tbody>
<tr>
<td>1</td>
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Extra forward
Granular media
Granular media

Barchan dunes

Aeolian sand transport
Splash experiment in Rennes

Oger & Valance

Apparatus
Splash experiment in Rennes

Oger & Valance

Apparatus

Coefficient of restitution

Mean restitution coefficient

Impact angle $\theta_1$ (degree)
Splash experiment in Rennes

Splash of particles

\[ V' = 2 \sqrt{gd}, \text{ all } V_i, \theta_i \]

# ejected \( \propto V_i \)
Splash experiment in Rennes

Splash of particles

\[ V' = 2\sqrt{gd}, \text{ all } V_i, \theta_i \]
Splash experiment in Rennes

Splash of particles

\[ V' = 2\sqrt{gd}, \text{ all } V_i, \theta_i \]

\# ejected \propto V_i
Spin in billiards

Before

\[ V = 1 \quad 0 \]
Spin in billiards

Before

\[ V = 1 \quad 0 \]

After

\[ 0 \quad 1 \]
Spin in billiards

Before

After

\[ V = 1 \quad 0 \]

\[ 0 \quad 1 \]

But spinning when rolling
Spin in billiards

Before

\[ V = 1 \quad 0 \]
\[ \omega = -1 \quad 0 \]
Spin in billiards

Before

After if $\mu > 0.2$

$V = 1 \quad 0$

$\omega = -1 \quad 0$

$0 \quad 1$

$-0.5 \quad 0.5$
Spin in billiards

Before

\[ V = 1 \quad 0 \]
\[ \omega = -1 \quad 0 \]

After if \( \mu > 0.2 \)

\[ 0 \quad 1 \]
\[ -0.5 \quad 0.5 \]

Later, rolling again

\[ \frac{1}{7} \quad \frac{4}{7} \]
\[ -\frac{1}{7} \quad -\frac{4}{7} \]
Spin in billiards

Before

\[ V = 1 \quad 0 \]

\[ \omega = -1 \quad 0 \]

\[ \text{Energy} \quad \frac{1}{2} mV^2 = \frac{7}{5} \]

After if \( \mu > 0.2 \)

\[ 0 \quad 1 \]

\[ -0.5 \quad 0.5 \]

\[ \frac{6}{5} \quad (86\%) \]

Later, rolling again

\[ \frac{1}{7} \quad \frac{4}{7} \]

\[ -\frac{1}{7} \quad -\frac{4}{7} \]

\[ \frac{17}{35} \quad (35\%) \]
Spin in billiards

Before

\[ V = 1 \quad 0 \]
\[ \omega = -1 \quad 0 \]

After if \( \mu > 0.2 \)

\[ 0 \quad 1 \]
\[ -0.5 \quad 0.5 \]

Later, rolling again

\[ \frac{1}{7} \quad \frac{4}{7} \]
\[ -\frac{1}{7} \quad -\frac{4}{7} \]

\[
\frac{\text{Energy}}{\frac{1}{2}mV^2} = \frac{7}{5} \quad \frac{6}{5} \quad \frac{17}{35}
\]

(86\%) \quad (35\%)

Energy lost through spinning rather than restitution
Newton’s cradle

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Extra forward