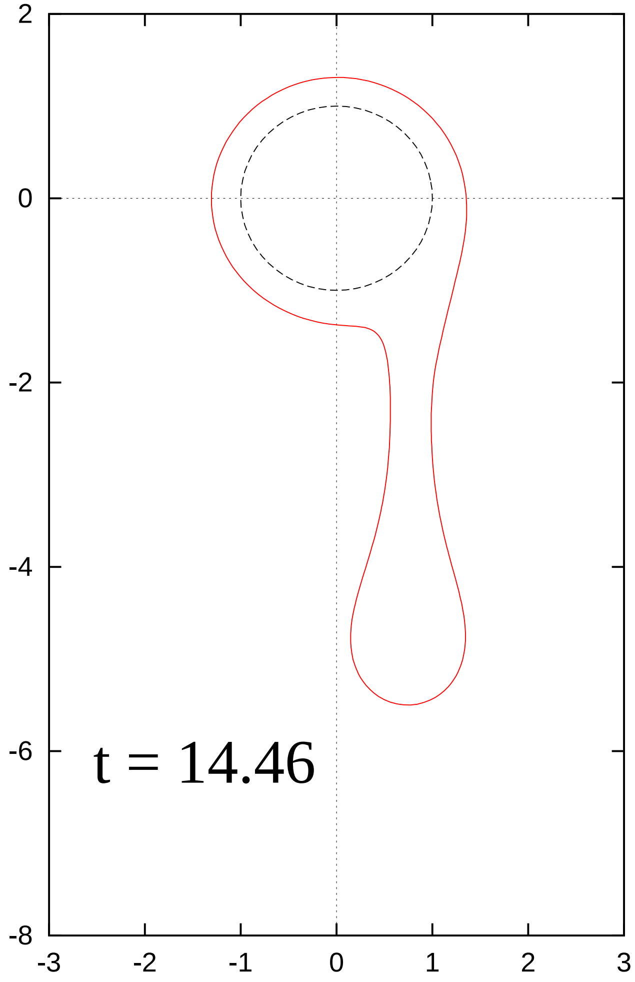
Shocking, the slow changes in HKM's thin coat

by HKM2 = Hinch, Kelmanson, Metcalfe (27 years later)

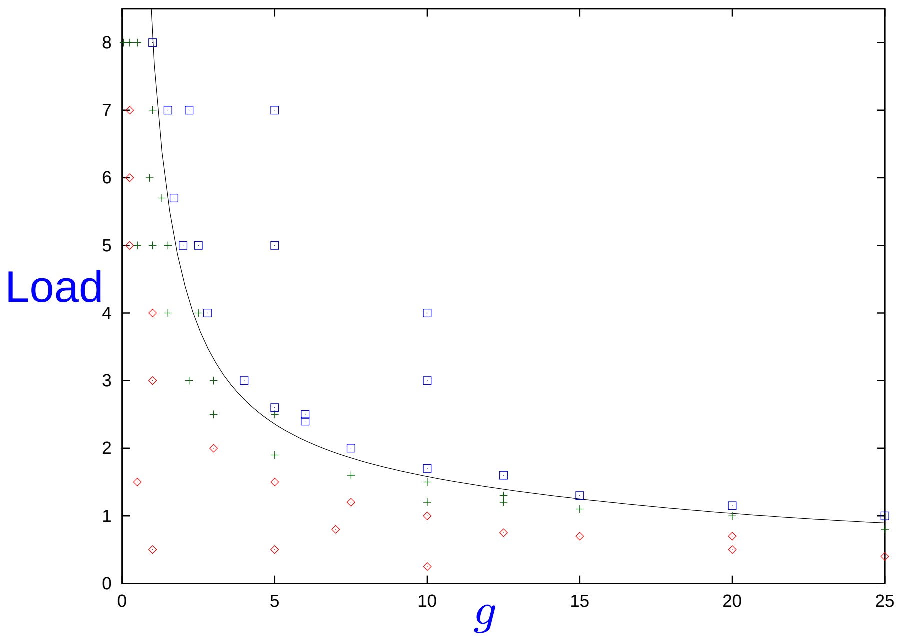
- Coating of a viscous liquid on the outside of a horizontal rotating cylinder
- Steady solution (for maximum load):
  - H.K. Moffatt (1977) *J. Fluid Mech.* 187
  - V.V. Pukhnachev (1977) *Z. Prik. Mekh. Tekh. Fiz* 3
- Transients?

Dripping if load too large



FEM solution by Peterson, Jimack & Kelmanson 2001

Curious oscillations



Drip □, steady +, oscillate ◊

FEM solutions by Peterson, Jimack & Kelmanson 2001

Or slowly decaying?

Lubrication theory (2D)

Mass conservation

$$h_t + q_x = 0$$

Flux

$$q = \omega a h - \frac{h^3}{3\mu}\left(\rho g \cos\theta - p_x\right)$$

Capillary pressure

$$p = \sigma(h_{xx} + h/a^2)$$

with $x = a\theta$.

Initial condition: $h(\theta, t = 0) = h_0$, uniform thickness.

Naive expansion (page 1 of 8)

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17-term naive expansion solution of Hinch & Kelmanson (2002) eqn (2.4)

$$Hhuge := h0 + h0^3\left(\frac{1}{3}\cos(\theta)\,\gamma0 - \frac{1}{3}\cos(\theta - t)\,\gamma0\right)$$
$$+ h0^5\left(\frac{1}{6}\cos(2\,\theta)\,\gamma0^2 - \frac{1}{3}\cos(2\,\theta - t)\,\gamma0^2 + \frac{1}{6}\cos(2\,\theta - 2\,t)\,\gamma0^2\right) + h0^7\left(-\frac{1}{12}\cos(\theta - t)\,\gamma0^3\right.$$
$$+ \frac{1}{9}\cos(3\,\theta)\,\gamma0^3 - \frac{1}{9}\cos(\theta - 2\,t)\,\gamma0^3 + \frac{1}{3}\cos(3\,\theta - 2\,t)\,\gamma0^3 - \frac{1}{3}\cos(3\,\theta - t)\,\gamma0^3 + \frac{2}{9}\cos(\theta)\,\gamma0^3$$
$$\left. - \frac{1}{36}\cos(\theta + t)\,\gamma0^3 - \frac{1}{9}\cos(3\,\theta - 3\,t)\,\gamma0^3 + \frac{5}{18}\gamma0^3\sin(\theta - t)\,t\right)$$
$$+ h0^8\left(-\frac{1}{3}\sin(2\,\theta)\,\alpha\,\gamma0^2 + \frac{4}{3}\sin(2\,\theta - t)\,\alpha\,\gamma0^2 - \sin(2\,\theta - 2\,t)\,\alpha\,\gamma0^2 - \frac{2}{3}\alpha\,\gamma0^2\cos(2\,\theta - 2\,t)\,t\right) +$$
$$h0^9\left(\frac{55}{108}\cos(4\,\theta - 2\,t)\,\gamma0^4 + \frac{55}{648}\cos(4\,\theta)\,\gamma0^4 + \frac{95}{324}\cos(2\,\theta)\,\gamma0^4 - \frac{55}{108}\cos(2\,\theta - t)\,\gamma0^4\right.$$
$$+ \frac{5}{18}\gamma0^4\,t\sin(2\,\theta - t) - \frac{55}{162}\cos(4\,\theta - t)\,\gamma0^4 + \frac{55}{324}\cos(2\,\theta - 3\,t)\,\gamma0^4 - \frac{5}{162}\cos(2\,\theta + t)\,\gamma0^4$$
$$\left. + \frac{55}{648}\cos(4\,\theta - 4\,t)\,\gamma0^4 - \frac{25}{54}\gamma0^4\sin(2\,\theta - 2\,t)\,t - \frac{55}{162}\cos(4\,\theta - 3\,t)\,\gamma0^4 + \frac{25}{324}\cos(2\,\theta - 2\,t)\,\gamma0^4\right) +$$
$$h0^{10}\left(-\frac{5}{12}\gamma0^3\sin(\theta - t)\,\alpha - \frac{11}{9}\gamma0^3\sin(3\,\theta)\,\alpha - 10\,\gamma0^3\sin(3\,\theta - 2\,t)\,\alpha + \gamma0^3\sin(\theta - 2\,t)\,\alpha\right.$$
$$- \gamma0^3\,\alpha\,t\cos(3\,\theta - 2\,t) + \frac{1}{3}\gamma0^3\,\alpha\,t\cos(\theta - 2\,t) + \frac{19}{6}\gamma0^3\,\alpha\cos(3\,\theta - 3\,t)\,t + \frac{197}{36}\gamma0^3\sin(3\,\theta - 3\,t)\,\alpha$$
$$\left. + \frac{7}{6}\gamma0^3\,\alpha\cos(\theta - t)\,t + \frac{23}{4}\gamma0^3\sin(3\,\theta - t)\,\alpha + \frac{1}{12}\gamma0^3\sin(\theta + t)\,\alpha - \frac{2}{3}\gamma0^3\sin(\theta)\,\alpha\right) + h0^{11}\Bigg($$
$$- \frac{455}{1296}\cos(5\,\theta - t)\,\gamma0^5 - \frac{91}{432}\cos(3\,\theta - 4\,t)\,\gamma0^5 + \frac{455}{1296}\cos(5\,\theta - 4\,t)\,\gamma0^5 + \frac{455}{648}\cos(5\,\theta - 2\,t)\,\gamma0^5$$
$$+ \frac{5}{162}\cos(3\,\theta - 3\,t)\,\gamma0^5 + \frac{1}{648}\cos(\theta + 2\,t)\,\gamma0^5 - \frac{115}{1296}\cos(\theta + t)\,\gamma0^5 - \frac{7}{216}\cos(3\,\theta + t)\,\gamma0^5$$
$$+ \frac{91}{1296}\cos(5\,\theta)\,\gamma0^5 - \frac{265}{648}\cos(\theta - 2\,t)\,\gamma0^5 + \frac{5}{9}\gamma0^5\,t\sin(\theta - t) + \frac{5}{18}\gamma0^5\,t\sin(\theta - 2\,t)$$
$$- \frac{5}{6}\gamma0^5\,t\sin(3\,\theta - 2\,t) - \frac{5}{216}\gamma0^5\,t\sin(\theta + t) + \frac{5}{18}\gamma0^5\,t\sin(3\,\theta - t) - \frac{14}{3}\cos(2\,\theta - 2\,t)\,\alpha^2\,\gamma0^2$$
$$+ 4\,\gamma0^2\,\alpha^2\sin(2\,\theta - 2\,t)\,t + \frac{4}{3}\gamma0^2\,\alpha^2\cos(2\,\theta - 2\,t)\,t^2 + \frac{65}{108}\gamma0^5\sin(3\,\theta - 3\,t)\,t + \frac{25}{216}\gamma0^5\cos(\theta - t)\,t^2$$
$$- \frac{91}{1296}\cos(5\,\theta - 5\,t)\,\gamma0^5 - \frac{2}{3}\cos(2\,\theta)\,\alpha^2\,\gamma0^2 + \frac{16}{3}\cos(2\,\theta - t)\,\alpha^2\,\gamma0^2 - \frac{205}{216}\cos(3\,\theta - t)\,\gamma0^5$$

Four time scales

- Rotation with cylinder $1/\omega$
- Surface tension makes envelope cylindrical $\mu a^4/\sigma h^3$
- Slow drift of rotation $\omega\mu^2 a^2/\rho^2 g^2 h^4$
- Slow decay of wobble $\omega^2\mu^3 a^6/\rho^2 g^2 \sigma h^7$.

K: 'Why weeny water waves wobble, wane & wander whilst whirling'

Nondimensionalise

To solve

$$h_t + h_\theta - \gamma(h^3 \cos\theta)_\theta + \alpha(h^3(h_{\theta\theta} + h)_\theta)_\theta = 0$$

with $h(\theta, 0) = 1$.

gravity $\quad \gamma = \dfrac{\rho g h^2}{3\mu\omega a}$

surface tension $\quad \alpha = \dfrac{\sigma h^3}{3\mu\omega a^4}$

Multiple scales 1 (easy)

To solve

$$h_t + h_\theta - \gamma(h^3 \cos\theta)_\theta + \alpha(h^3(h_{\theta\theta} + h)_\theta)_\theta = 0$$

with $h(\theta, 0) = 1$, for $\gamma \ll 1$ with $\alpha = O(1)$ small.

Expand

$$h \sim 1 + \gamma h_1 + \gamma^2 h_2 + \gamma^3 h_3$$

with

$$h_n(\theta, \tau = t, T = \gamma^2 t)$$

Multiple scales 1, $O(\gamma)$

$O(\gamma)$:

$$h_{1\tau} + h_{1\theta} + \alpha(h_{1\theta\theta\theta\theta} + h_{1\theta\theta}) = -\sin\theta$$

Solution

$$h_1 = \cos\theta + A(T)\cos(\theta - \tau) + B(T)\sin(\theta - \tau)$$

$$+ \sum_{n>1} \left[A_n \cos n(\theta - \tau) + B_n \sin n(\theta - \tau)\right] e^{-\alpha n^2 (n^2 - 1)\tau}$$

with $A(0) = -1$ and $B(0) = 0$.

This is steady solution + eccentrically rotating circle + surface tension squeezing to a circle.

Multiple scales 1, $O(\gamma^2)$

$O(\gamma^2)$:

$$h_{2\tau} + h_{2\theta} + \alpha(h_{2\theta\theta\theta\theta} + h_{2\theta\theta}) = -(3h_1 \cos\theta)_\theta$$

Solution – a mess

$$h_2 = \frac{3}{2(1+36\alpha^2)}\cos 2\theta - \frac{9\alpha}{1+36\alpha^2}\sin 2\theta + \frac{3(A+12\alpha B)}{1+144\alpha^2}\cos(2\theta-\tau) + \frac{3(B-12\alpha A)}{1+144\alpha^2}\sin(2\theta-\tau)$$

after others decay by surface tension.

Multiple scales 1, $O(\gamma^3)$

$O(\gamma^3)$:

$$h_{3\tau} + h_{3\theta} + \alpha(h_{3\theta\theta\theta\theta} + h_{3\theta\theta})$$
$$= -h_{1T} - ((3h_2 + 3h_1^2)\cos\theta)_\theta - \alpha(3h_1(h_{2\theta\theta} + h_2)_\theta)_\theta$$

Secularity condition – resonant $\sin(\theta - \tau)$ and $\cos(\theta - \tau)$

$$\frac{d}{dT}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

where

$a = 81\alpha/(1 + 144\alpha^2)$ and $b = 3(5 + 72\alpha^2)/2(1 + 144\alpha^2)$

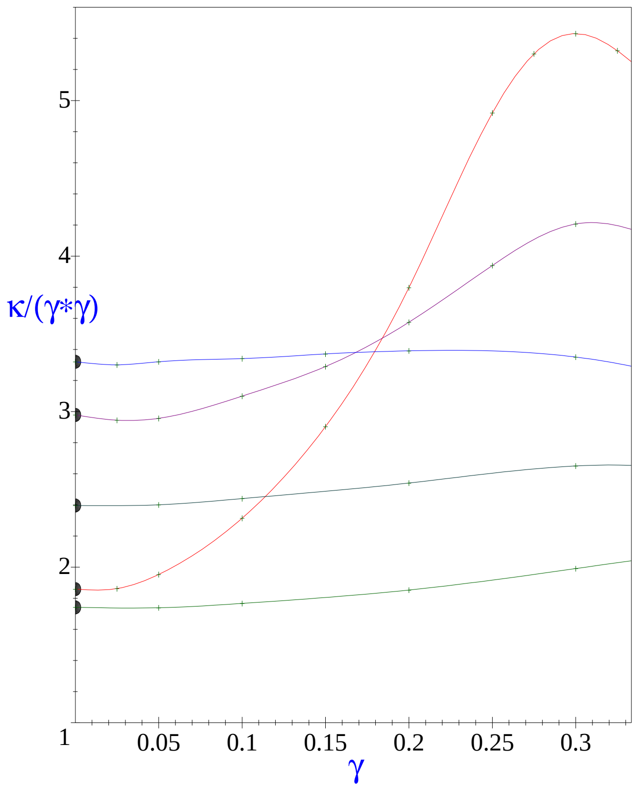
Multiple scales 1, decay rate

Hence decay rate

$$\kappa = 81\gamma^2\alpha/(1+144\alpha^2)$$

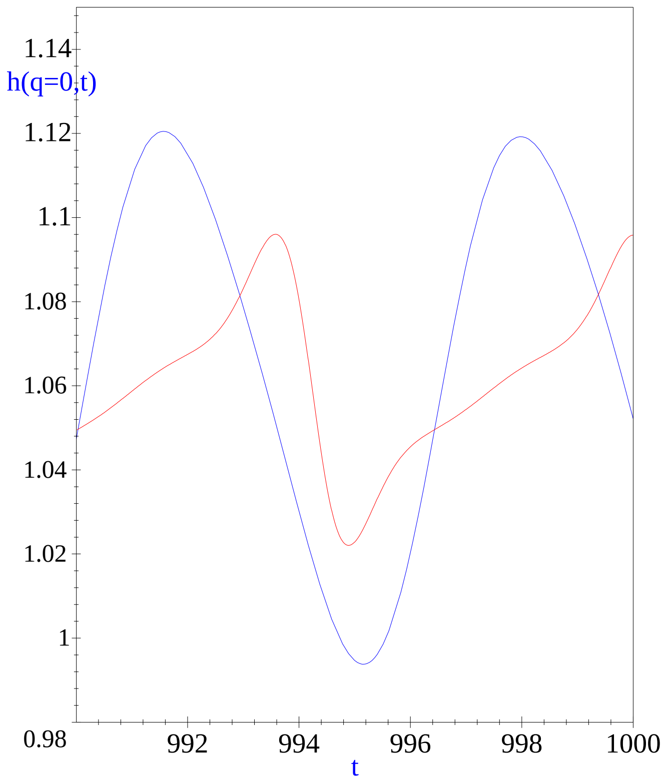
$\alpha =$ 0.025, 0.05, 0.1, 0.2, 0.3

$$[\gamma^2\alpha = \omega^2\mu^3a^6/\rho^2g^2\sigma h^7]$$



- Maximum load becomes steady in 3 revolutions, half-maximum in 300 and quarter-maximum in 3000, because $\kappa \propto h^{-7}$.

M's problem



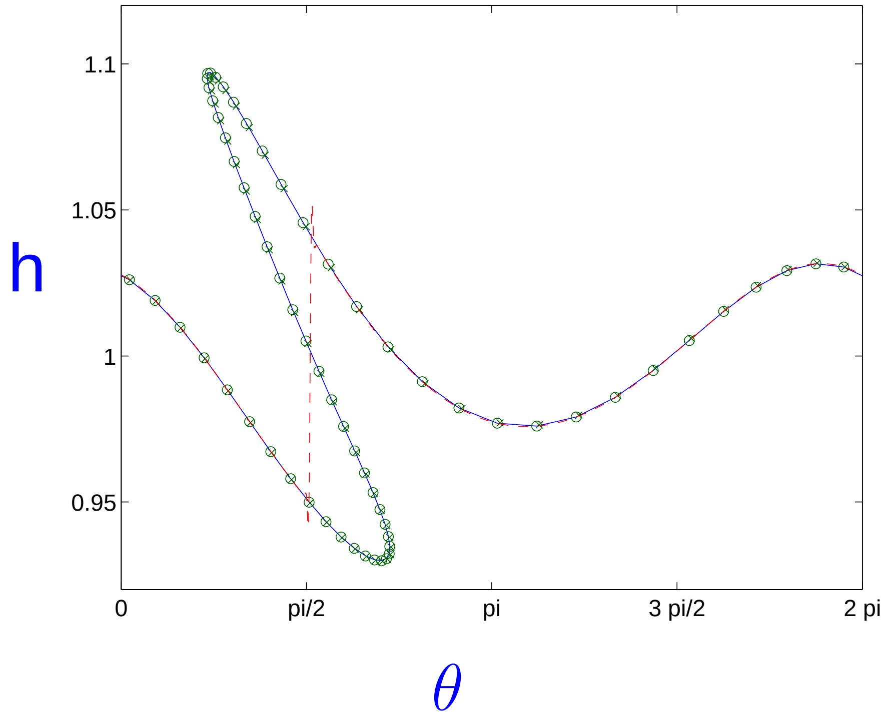
$h(\theta = 0, t)$ for

$\gamma = 5.32\,10^{-2}$, $\alpha = 4.8\,10^{-5}$

Profile not
theoretical $\cos + \cos 2$
but shocking!

First study valid for $\gamma^2 \ll \alpha, \gamma \ll 1$ What if $\alpha \ll \gamma^2$?

No surface tension



$t = 500$

$\alpha = 0$ ——— numerical, ○, × action-angle integration

$\alpha = 10^{-9}$ - - - - - numerical (spectral)

Multiple scales 2

To solve

$$h_t + h_\theta - \gamma(h^3\cos\theta)_\theta + \alpha(h^3(h_{\theta\theta} + h)_\theta)_\theta$$

for $\gamma \ll 1$ and $\beta = \alpha/30\gamma^3 = O(1)$.

Expand

$$h \sim 1 + \gamma h_1 + \gamma^2 h_2 + \gamma^3 h_3 + \gamma^4 h_4$$

with $h_n(\theta, \tau, T)$

$$\tau = t\left(1 - \frac{15}{2}\gamma^2 - \frac{1215}{8}\gamma^4\right) \quad \text{and} \quad T = 30\gamma^3 t$$

Multiple scales 2, $O(\gamma)$

$O(\gamma)$:

$$h_{1\tau} + h_{1\theta} = -\sin\theta$$

Solution

$$h_1 = \cos\theta + F(\eta = \theta - \tau, T)$$

with

$$F(\eta, 0) = -\cos\eta$$

Disturbance rotating with cylinder. To find evolution.

Multiple scales 2, $O(\gamma^2)$

$O(\gamma^2)$:

$$h_{2\tau} + h_{2\theta} = (3h_1 \cos\theta)_\theta$$

Solution

$$\begin{aligned} h_2 = \quad & 3F\cos\theta + 3F'\sin\theta \\ & + \text{steady state} + G(\eta, T) \end{aligned}$$

Multiple scales 2, $O(\gamma^3)$

$O(\gamma^3)$:

$$h_{3\tau} + h_{3\theta} = \left(3(h_2 + h_1^2)\cos\theta\right)_\theta$$

Solution

$$\begin{aligned} h_3 = \quad & \tfrac{15}{2} F \cos 2\theta + 3F^2 \cos\theta + \tfrac{33}{4} F' \sin\theta - \tfrac{9}{4} F'' \cos 2\theta \\ & + \text{steady state} + \text{ terms G} + H(\eta, T) \end{aligned}$$

Multiple scales 2, $O(\gamma^4)$

$O(\gamma^3)$:

$$\begin{aligned} h_{4\tau} + h_{4\theta} = \quad & -h_{1T} - 30\beta\left(h_{1\theta\theta} + h_1\right)_{\theta\theta} \\ & + \left((3h_3 + 6h_2h_1 + h_1^3)\cos\theta\right)_\theta \end{aligned}$$

Secularity : Kuramoto-Sivashinsky equation

$$F_T - FF_\eta + \beta(F_{\eta\eta\eta\eta} + F_{\eta\eta}) = 0$$

with

$$F(\eta, 0) = -\cos\eta$$

Governing equation now has constant coefficients

KS with $\beta \ll 1$

$$F_T - F F_\eta + \beta(F_{\eta\eta\eta\eta} + F_{\eta\eta}) = 0 \quad \text{with} \quad F(\eta, 0) = -\cos\eta$$

Set $\beta = 0$.

Shock forms at

$$T = 1 \quad \text{where} \quad \eta = \frac{\pi}{2}$$

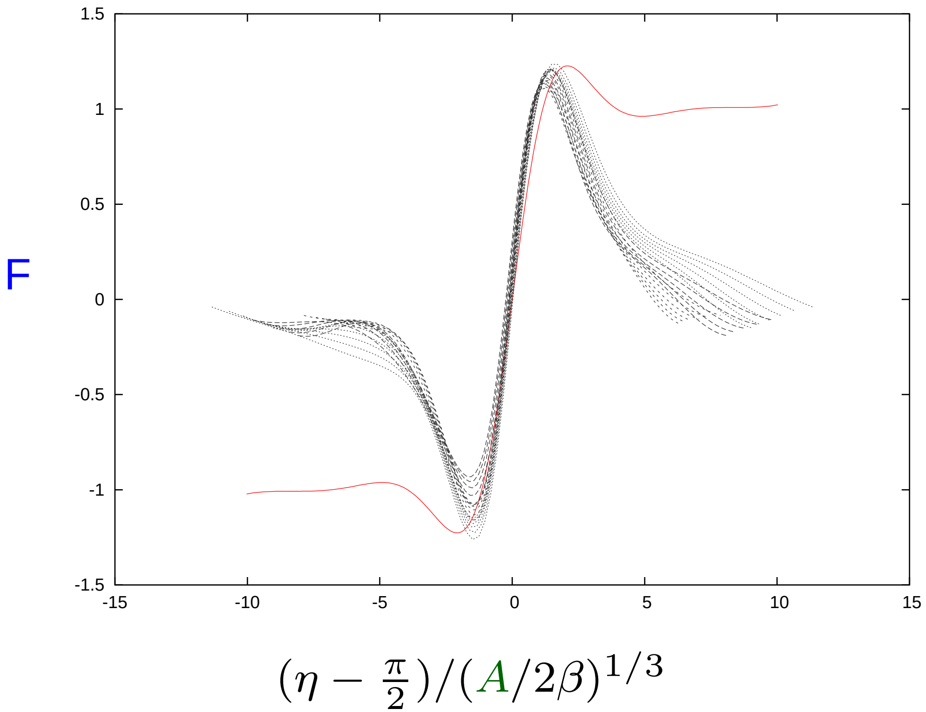
Afterwards, decay of a N-shock wave

$$F \sim \frac{\frac{\pi}{2} + \eta}{T}$$

Structure of the shock

Quasi-steady, with $F \to \pm A(T)$ outside, $A = \pi/T$

$$F^2 - A^2 = 2\beta F'''$$



at different times.
for $\gamma = 0.1, 0.08$
and $\alpha = 10^{-4}, 3\,10^{-4}$

But width $(2\beta/A)^{1/3}$ no longer thin when $T = 1/\beta$.

KS with $\beta \gg 1$

Multiple scales 3.   Slow-slow time $\mathtt{T} = \beta^{-1} T$

$$F \sim A(\mathtt{T}) \cos \eta - \frac{1}{24} \beta^{-1} A^2 \sin 2\eta$$

$O(\beta^{-2})$: secularity

$$A_{\mathtt{T}} = \frac{1}{48} A^3 \quad \text{with} \quad A(0) = -1$$

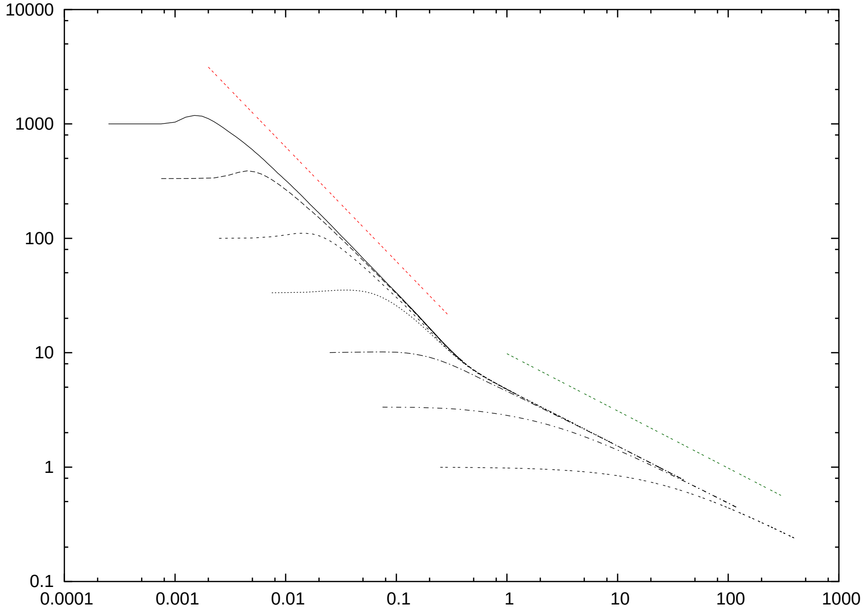
Solution

$$A = -\left(1 + \frac{\mathtt{T}}{24}\right)^{-1/2}$$

Can also use for $\beta \ll 1$ once $\beta T \gg 1$.

Decay of KS solutions

$\frac{F_{\max}}{\beta}$



βT

for 7 β: 0.003 to 1

Decay: T^{-1}, then $T^{-1/2}$, and then ...

…and then

After algebraic decay of KS equation, eventually exponential decay of lubrication equation

$$F_{\max} \sim \frac{\alpha}{\left(\alpha^2 + \frac{25}{108}\gamma^4\right)^{1/2}} e^{-81\alpha\gamma^2 t}$$

NB: first study needed $\alpha \gg \gamma^2$.

Conclusions

- Transients decay
  - $\gamma^2 \ll \alpha, \gamma \ll 1$: exponentially
  - $\gamma^3 \ll \alpha \ll \gamma^2$: $T^{-1/2} \rightarrow$ exponentially
  - $\gamma^4 \ll \alpha \ll \gamma^3$: $T^{-1} \rightarrow T^{-1/2} \rightarrow$ exponentially
- HKM's steady state much simpler!