Inclined to exchange: shocking gravity currents

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Experimental setup

Dense, transparent solution ($\rho_2$)

Digital camera

Tube diameter $d = 20\text{mm}$
Tube length : $3.5\text{ m}$

Light, dyed solution ($\rho_1$)

Gate valve

Video camera

$At = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}$

Water-CaCl$_2$ solutions : $\text{CaCl}_2 : 1$ to $100\text{ g/l} \Rightarrow At = 4 \times 10^{-4} - 3.5 \times 10^{-2}$
Regimes

- Vertical: turbulent
- Inclined: nosed controlled gravity current
- Horizontal: viscous counter-current
Speed of front $V_f$

\[ V_t = \sqrt{Atgd} \quad \text{(inertial nose)} \quad V_\nu = \frac{Atgd^2}{\nu} \quad \text{(viscous)} \]

Tilt angles $0 < \alpha < 30^\circ$;

$At = 3.5 \times 10^{-2}$ (●), $10^{-2}$ (■), $4 \times 10^{-3}$ (♦), $10^{-3}$ (▼), $4 \times 10^{-4}$ (▲) for a viscosity $\mu = 10^{-3} \text{ Pa.s}$;

and $\mu = 10^{-3} \text{ Pa.s}$ (■), $4 \times 10^{-3} \text{ Pa.s}$ (▩) for a density contrast $At = 10^{-2}$

\[ V_f \approx 0.014V_\nu \sin \alpha \]
A little theory

Counter current in a circle at 50% fill

\[ \nabla^2 u = f(\theta) = \begin{cases} 
-\frac{1}{2} & \text{in } 0 \leq \theta < \pi, \\
+\frac{1}{2} & \text{in } \pi \leq \theta < 2\pi 
\end{cases} \]

and \( u = 0 \) on \( r = 1 \).

Fourier series

\[ f(\theta) = -\sum_{n \text{ odd}} \frac{2}{\pi n} \sin n\theta, \]

so

\[ u = \sum_{n \text{ odd}} \frac{2}{\pi n(n^2 - 4)}(r^2 - r^n) \sin n\theta. \]
Hence flux

\[ Q = \sum_{n \text{ odd}} \frac{1}{\pi n^2 (n + 2)^2} \]

\[ = \frac{\pi}{16} - \frac{1}{2\pi} \]

\[ = 0.037195 \]

Front velocity for area \( A = \frac{\pi}{2} \)

\[ V_f = \frac{Q}{A} = 0.012, \]

experiment 0.014 – eventually will do better.
\[ \dot{X}_f = V_f \]
\[ \propto \frac{R^2 \, dp}{\mu \, dx} \]
\[ \propto \frac{R^2 \, \Delta \rho g R}{\mu \, X_f} \]

Hence

\[ X_f = \sqrt{Dt} \]

\[ D = 0.0108 V_\nu d \text{ in experiment} \]
2D spreading in a horizontal channel

\[
\mu \frac{\partial^2 u}{\partial y^2} = \frac{dp_\pm}{dx} \quad \text{in } 0 \leq y \leq h \text{ and } h \leq y \leq a,
\]

with \( u = 0 \) on \( y = 0 \) and \( a \).

No net flux \( \int u \, dy = 0 \) and \[
\left[ \frac{dp}{dx} \right]^+ = \Delta \rho g \frac{\partial h}{\partial x}.
\]

Hence

\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\Delta \rho gh^3(a-h)^3}{3\mu a^3} \frac{\partial h}{\partial x} \right).
\]
Spreading in a horizontal *channel*, continued

Hence

\[ X_f = \sqrt{D t} \]

with \( D = 0.017V \nu a \), cf experiment 0.0108.
Spreading in a horizontal circle

Flat interface at \( y = h(x, t) \). Flow in cross-section

\[
\nabla^2 u = G + \begin{cases} 
-\frac{1}{2} & \text{in } h \leq y \leq 1, \\
+\frac{1}{2} & \text{in } -1 \leq y < h
\end{cases}
\]

and \( u = 0 \) on \( r = 1 \). \( G \) so no net flux.

Numerical (finite difference 80 × 80, sor) for flux \( Q(h) \) in \( h \leq y \leq 1 \).
Thin layer asymptotics, $1 - |h| \ll 1$

No net flow $\Rightarrow$ all pressure gradient in thin layer.
Thin layer $\Rightarrow$ no stress by thick on thin.
Wide compared with thin $\Rightarrow$ 2D, half parabolic profile.

$$Q(h) \sim \frac{32\sqrt{2}}{105} (1 - |h|)^{7/2}.$$ 

Symmetric and smooth $Q(h)$, so equally

$$Q(h) \sim \frac{4}{105} (1 - h^2)^{7/2}.$$
Lucky break

Thin layer asymptotic's \( \frac{4}{105} = 0.038095 \)

50% fill \( Q(0) = 0.037195 \)

Hence good approximation (within 1\%):

\[
Q(h) \approx Q(0)(1 - h^2)^{7/2}
\]
Nonlinear diffusion equation

\[ \frac{\partial A(h)}{\partial t} + \frac{\partial Q(h)}{\partial x} = 0, \]

with cross-sectional area \( A(h) = 2 \cos^{-1} h - h \sqrt{1 - h^2} \), so \( dA/dh = -2\sqrt{1 - h^2} \). Hence

\[ \frac{\partial h}{\partial t} = \frac{1}{2\sqrt{1 - h^2}} \frac{\partial}{\partial x} \left( (1 - h^2)^{7/2} \frac{\partial h}{\partial x} \right). \]

\[ X_f = \sqrt{Dt} \]

with \( D = 0.0108V_v d, \)
experiment 0.0108 ± 0.0008.
Comparison with experiment
Spreading in nearly horizontal tubes

Flow now driven by difference in pressure gradients in two fluids

\[ \Delta \rho g \left( \cos \alpha \frac{\partial h}{\partial x} + \sin \alpha \right). \]

So with suitable rescaling

\[ \frac{\partial h}{\partial t} = \frac{1}{2\sqrt{1 - h^2}} \frac{\partial}{\partial x} \left( (1 - h^2)^{7/2} \left( \frac{\partial h}{\partial x} + 1 \right) \right). \]

\[ t = 0.5, (0.5), 5.0. \]
Diffusive spreading only at very short times.
Slow to attain long-time velocity.
Experimental motion of front \( X_f(t) \)

\[
2 X_f / (d \cot \alpha) = \frac{t \left( 8 F(0) V_y \sin^2 \alpha \right)}{(d \cos \alpha)}
\]

\( \alpha = 1^\circ \circ, \)

\( 2^\circ \bigtriangleup, \)

\( 3^\circ \bigtriangledown. \)

Experiments straighter due to different early behaviour – gate release and a short time limited by inertia.
Long time approximation?

\[ X_f(t) \sim \begin{align*}
0.856t + 1.01 & \quad 4 < t < 5 \\
0.830t + 1.19 & \quad 7 < t < 11 \\
0.774t + 1.83 & \quad 20 < t < 25 \\
0.750t + 2.82 & \quad 90 < t < 100 \\
0.746t + 3.36 & \quad 145 < t < 150
\end{align*} \]

Linear fit fails.
Logarithms at long time

Crude model: horizontal + inclined spreading

\[
\dot{X}_f = \frac{V_f L}{X_f} + V_f,
\]

with solution

\[
V_f t = X_f - L \log\left(1 + \frac{X_f}{L}\right).
\]

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<th>(L)</th>
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Slow logarithmic approach also in experimental data.
Test $V_f = 0.742$

Continuous line is new $0.742 \ (0.014V_\nu \sin \alpha)$, dashed is old $0.637 \ (0.012V_\nu \sin \alpha)$. 
But why $V_f = 0.742$?

$$\frac{\partial h}{\partial t} = \frac{1}{2\sqrt{1 - h^2}} \frac{\partial}{\partial x} \left( (1 - h^2)^{7/2} \left( \frac{\partial h}{\partial x} + 1 \right) \right).$$

For $h_x \ll 1$

$$\frac{\partial h}{\partial t} + c(h) \frac{\partial h}{\partial x} = 0 \quad \text{with kinematic wave speed} \quad c(h) = \frac{7}{2} h (1 - h^2)^{2}.$$ 

Test rarefaction wave solution by plotting $h$ vs $x/t$.

But must shock
Shock waves

Speed of shock \( V(h) = \frac{Q(h)}{A(h)} \)

must equal speed of arriving characteristics \( c(h) \).

Equal at \( h = 0.23817 \) with \( c = V = 0.74172 \).
Form of shock waves

Switch to frame moving with $V$, then steady form governed by

$$\frac{\partial h}{\partial x} = V \frac{A(h)}{Q(h)} - 1$$
Conclusions

- Front is a shockwave:
  speed selected by matching onto a rarefaction wave.
- Logarithmic approach to long-time:
  value depends on initial condition.
And next

- Vertical: turbulent
- Inclined: nosed controlled gravity current
- Horizontal: viscous counter-current