Boundary Integral/Element Method

- For linear problems with known simple Greens functions
  e.g. potential flows, Stokes flows
- Good for complex geometry
- Very good for free surface problems needing only $u$ on the surface

Greens identity (divergence theorem)

$$
\int_S \left( \phi \frac{\partial G}{\partial n} - \frac{\partial \phi}{\partial n} G \right) \, dS(x) = \int_V \left( \phi \nabla^2 G - \nabla^2 \phi \, G \right) \, dV(x)
$$

$$
= \int_V \phi(x) \delta(x - \xi) \, dV(x)
$$

$$
= \phi(\xi) \times \begin{cases} 
0 & \text{if } \xi \text{ outside } V, \\
1 & \text{if } \xi \text{ inside } V, \\
\frac{1}{2} & \text{if } \xi \text{ on smooth } S, \\
\frac{1}{4} \Omega & \text{if } \xi \text{ at corner of } S \text{ with solid angle } \Omega. 
\end{cases}
$$

Laplace equation

$$
\nabla^2 \phi = 0 \quad \text{in the volume } V
$$

$$
\phi \quad \text{or} \quad \frac{\partial \phi}{\partial n} \quad \text{given on the surface } S
$$

where $n$ the unit normal to the surface out of the volume.

Need Greens function $G(x, \xi)$, viewing $\xi$ as a fixed parameter

$$
\nabla_x G = \delta(x - \xi) \quad \text{for } x \text{ in } V
$$

$G$ need not satisfy any BC on $S$

$$
\nabla_x \text{ means differentiate with respect to } x
$$

Boundary integral equation

For $\xi$ on smooth $S$

$$
\frac{1}{2} \phi(\xi) = \int_S \left( \phi \frac{\partial G}{\partial n} - \frac{\partial \phi}{\partial n} G \right) \, dS(x)
$$

Either given $\phi|_S$, solve for $\frac{\partial \phi}{\partial n}|_S$

Or given $\frac{\partial \phi}{\partial n}|_S$, solve for $\phi|_S$

Then find $\phi$ inside $V$ by evaluation integral with $1$ replacing $\frac{1}{2}$

For exterior problem, add $\phi_\infty(\xi)$ to RHS of integral equation
Greens functions

Normally use ‘free-space’ Greens functions

\[ G = -\frac{1}{4\pi |x - \xi|}, \quad \frac{\partial G}{\partial n} = \frac{(x - \xi) \cdot n(x)}{4\pi |x - \xi|^3} \]

in \( R^3 \):

and in \( R^2 \):

\[ G = \frac{1}{2\pi} \ln |x - \xi|, \quad \frac{\partial G}{\partial n} = \frac{(x - \xi) \cdot n(x)}{2\pi |x - \xi|^2} \]

Become elliptic functions for axisymmetric

Sometimes use images so \( G \) satisfies BCs (simple geometries)

Integrand is singular

For fixed \( \xi \) on \( S \) and \( x \) moving on \( S \)

\[ G \propto \frac{1}{|x - \xi|} \text{ in } R^3, \quad G \propto \ln |x - \xi| \text{ in } R^2. \]

Integrable but singular – take care numerically

On smooth \( S \)

\[ n(x) \cdot (x - \xi) \sim \frac{1}{2} \kappa |x - \xi|^2, \]

where \( \kappa \) is the curvature. Hence

\[ \frac{\partial G}{\partial n} \sim \frac{\kappa}{8\pi |x - \xi|} \text{ in } R^3, \quad G \propto \frac{\kappa}{4\pi} \text{ in } R^2. \]

So no more singular

Hence need numerically smooth \( S \)

Eigensolutions

Interior problem has one eigensolution

\[ \phi = 1 \quad \text{and} \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } S \]

corresponding to

\[ \phi(x) = 1 \quad \text{in } V \]

Associated constraint

\[ \int_S \frac{\partial \phi}{\partial n} dS = 0 \]

from zero volume sources in \( \nabla^2 \phi = 0 \quad \text{in } V. \)

Discretise

1. Divided up \( S \) into ‘panels’
   - in \( R^2 \) a curve divided into segments
   - in \( R^3 \) normally triangles

2. Represent unknowns \( \phi \) and \( \frac{\partial \phi}{\partial n} \) by basis functions \( f_i(x) \) over the panels, e.g. piecewise constants/linear (or B-splines)

\[ \phi(x) = \sum \Phi_i f_i(x), \quad \frac{\partial \phi}{\partial n} = \sum D\Phi_i f_i(x) \]

with unknown amplitudes \( \Phi_i \) and \( D\Phi_i \).

3. Satisfy integral equation at collocation points
   or by least squares or with weighted integrals.

Suitable collocations points are:
- centre of panels for piecewise constant basis functions
- vertices of panels for piecewise linear basis functions.
Discretised integral equation

One thus forms a discretised version of the integral equation in terms of the amplitudes $\Phi_i$ and $D\Phi_i$

$$\left(\frac{1}{2}I - D\mathcal{G}\right)\Phi = -\mathcal{G}D\Phi,$$

where the matrix elements are

$$D\mathcal{G}_{ij} = \int_S f_j(x) \frac{\partial G}{\partial n}(x, \xi) \, dS(x), \quad \text{and} \quad \mathcal{G}_{ij} = \int_S f_i(x) G(x, \xi) \, dS(x),$$

both evaluated at $\xi = x_i$.

Evaluation of $\mathcal{G}$ and $D\mathcal{G}$

Short range integrals (if splines must use $B$-splines)

Often use Gaussian integration – avoids singular point $x = \xi$

Often use trapezoidal integration for $|i - j| > 3$ or $4$

Gaussian poor for self and next-to-self panels $|i - j| \leq 1$

8pt Gaussian → error $3 \times 10^{-15}$ in $\int_0^\pi \sin x$ x, but $9 \times 10^{-3}$ in $\int_0^\frac{1}{2} \ln x$

So subtract off the singularity and evaluate analytically

$$G(x, \xi) \sim a(\xi) \ln |x - \xi| + \text{regular term},$$

$$\int_{\xi - \delta_1}^{\xi + \delta_2} a(\xi) \ln |x - \xi| \, dx = a(\xi) (\delta_2 \ln \delta_2 - \delta_2 + \delta_1 \ln \delta_1 - \delta_1).$$

Regular term safely by the trapezoidal rule.
Similarly the next-to-self panel, if not one more beyond.

Avoiding eigensolution

Invert singular matrices

$$\left(\frac{1}{2}I - D\mathcal{G}\right)\Phi = -\mathcal{G}D\Phi,$$

in space orthogonal to eigensolution

Fix 1 Rely on truncation error to keep condition number finite

Fix 2 Make eigenvalue $\alpha$ rather than 0

$$A' = A + \alpha e e^\dagger$$

For interior problem

$$e = (1, 1, \ldots, 1) \quad \text{and} \quad (e^\dagger)_i = \int_S f_i \, dS$$

(so long as $\sum f_i(x) \equiv 1$)

Tests

In two dimensions

$$\phi = r^k \cos k\theta$$

with $\frac{\partial \phi}{\partial n} = n \cdot \nabla \phi = n_r k r^{k-1} \cos k\theta - n_\theta k r^{k-1} \sin k\theta$,

and similarly in three dimensions.

Test error is $O(\Delta x^2)$ if piecewise linear basis functions, and $O(\Delta x^4)$ if cubic splines
Costs

Boundary integral method has unknowns only on surface, so costs less?

- Volume method $N^2$ points in 2D, $N^3$ points in 3D
  - Fast Poisson solver (need regular geometry) $N \ln N$ steps
  - Cost $N^3 \ln N$ or $N^4 \ln N$

- Surface method $4N$ points in 2D, $6N^2$ points in 3D
  - Boundary integral method has dense matrix $\frac{1}{3}(\cdot)^3$ inversion
  - Costs $11N^3$ or $72N^6$

But BIM good for complex or $\infty$ geometry

Reduce cost to $(\cdot)^2$ by iteration from last time-step

Try Fast Multipoles

Free surface potential flows

Start time step with known surface $S(t)$ and potential $\phi(x, t)$ known on $S$
Use BIM to find $\partial \phi / \partial n$ on $S$, $\rightarrow \nabla \phi$
Evolve surface $\frac{Dx}{Dt} = \nabla \phi$ for points on $S$
Evolve surface potential $\frac{D\phi}{Dt} = \frac{1}{2} |\nabla \phi|^2 - \frac{g \cdot x}{\rho \kappa} - \rho_{atm}$ for points $x$ on $S$,

Capillary waves mean $\Delta t < \sqrt{\rho / \gamma} \Delta x^{3/2}$
A good test is the vibration frequencies of an isolated drop.
Problem: conserve energy $\rightarrow$ accumulate numerical noise in short capillary waves, so smooth or Fourier filter

Stokes flows

$$\frac{1}{2} u(\xi) = \int_S \left( (\sigma \cdot n) \cdot G - u \cdot K \cdot n \right) dS(x),$$

with the Greens function, called a Stokeslet, and its derivative

$$G = \frac{1}{8 \pi \mu} \left( \frac{1}{r} + \frac{rr}{r^3} \right) \quad \text{and} \quad K = -\frac{3}{4 \pi} \frac{rr}{r^5}, \quad \text{where} \quad r = x - \xi.$$

For drops, outside minus inside, so only need $[\sigma \cdot n] = -\gamma \kappa n$

$$\frac{1}{2} (\mu_{in} + \mu_{out}) u(\xi) = \int_S \left( [\sigma \cdot n] \cdot G - (\mu_{in} - \mu_{out}) u \cdot K \cdot n \right) dS(x),$$

Eigensolutions of rigid body motion for interior problem $- no motion from constant pressure