Resumé of lecture 2

Driven Cavity in ψ–ω formulation.

Finite Differences

Poisson problem. SOR.

Test against theoretical solution: $O(\Delta x^2)$ error?

2.7 Vorticity evolution

$$\frac{\partial \omega}{\partial t} = -\frac{\partial(\omega,\psi)}{\partial(x,y)} + \frac{1}{Re}\nabla^2\omega$$

with $\omega = 0$ at $t = 0$.

Forward time-step from $t = n\Delta t$ to $t = (n+1)\Delta t$
at interior points $i = 1 \to N-1$, $j = 1 \to N-1$

$$\omega_{ij}^{n+1} = \omega_{ij}^{n} + \Delta t \left[-\frac{\psi_{ij+1}^{n} - \psi_{ij-1}^{n}}{2\Delta x}\frac{\omega_{i+1j}^{n} - \omega_{i-1j}^{n}}{2\Delta x} + \frac{\psi_{i+1j}^{n} - \psi_{i-1j}^{n}}{2\Delta x}\frac{\omega_{ij+1}^{n} - \omega_{ij-1}^{n}}{2\Delta x} \right] + \frac{\Delta t}{Re\Delta x^2}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix}\omega_{ij}^{n}$$

On boundary need $\psi = 0$, and value of ω

Boundary condition on ω – so that $\frac{\partial \psi}{\partial n} = U_{\mathrm{wall}}$

For bottom $y = 0$:

$$u_{\frac{1}{2}} = \frac{\psi_{i1} - \psi_{i0}}{\Delta x}$$

so

$$\omega_{\frac{1}{4}} = \frac{u_{\frac{1}{2}} - U_{\mathrm{wall}}}{\frac{1}{2}\Delta x}$$

1st order BC

$$\omega_0 \approx \omega_{\frac{1}{4}} = \frac{\frac{\psi_{i1}-\psi_{i0}}{\Delta x} - U_{\mathrm{wall}}}{\frac{1}{2}\Delta x}$$

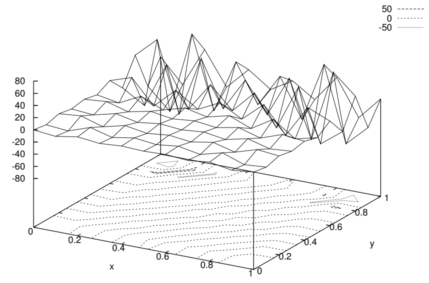
2nd order, by linear extrapolation

$$\omega_0 \approx \frac{4\omega_{\frac{1}{4}} - \omega_1}{3}.$$

Starts at $t = 0$ as numerical delta function, then diffuses.

2.8 Time-step instability

plot ω for $Re = 10$ at $t = 0.525$ with $\Delta t = 0.035$ and $\Delta x = 0.1$

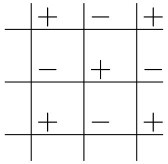


Numerical or physical instability?

Not physically unstable at $Re = 10$ surely?

Time step instability 2

Checker board pattern.



$$\omega_{ij}^n = (-)^{i+j} A_n,$$

Diffusion terms in time-stepping algorithm

$$A_{n+1} = A_n + \frac{\Delta t}{Re \Delta x^2} \cdot -8 A_n$$

Stable if $\Delta t < \frac{1}{4} Re \Delta x^2$ – at least one Δt to diffuse one Δx.

EJH works at $\frac{1}{5}$.

Advection instability → CFL condition

(Courant-Friedricks-Lewy)

Stable if $\Delta t < \Delta x / U_{\max}$ – at least one Δt to advect one Δx.

Must resolve boundary layers
Dimensional: $U_{\max} \Delta x / \nu < 1 \Leftrightarrow$ Nondimensional $\Delta x < \frac{1}{Re}$.
This + stable diffusion ⇒ stable advection

Total cost to $t = 1$

$$\left(\# \text{ time steps} \frac{1}{\Delta t} \propto N^2\right) \times \left(\text{cost per time step (SOR)} \propto N^3\right)$$

$$\propto N^5$$

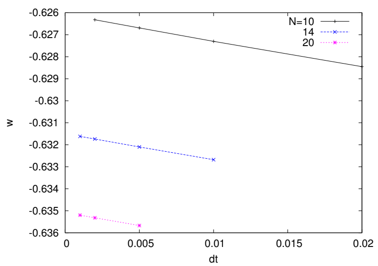
Hence doubling N is 32 times longer, quadruple N is 1024 longer.

'Better' time step algorithms → larger Δt, but more accurate?

2.9 Accuracy consistency. a. Time-stepping

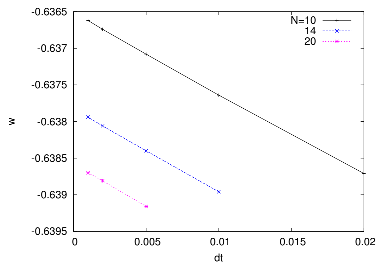
No analytic solution to test → test code has designed accuracy $O(\Delta t, \Delta x^2)$.

Forward differencing → $O(\Delta t)$ errors.
Look at $\omega(x = 0.5, y = 0.5, t = 1)$ – **exactly** $(0.5, 0.5, 1)$
1st order BC for ω_0 with $Re = 10$ and $N = 10$, 14 and 20.



Note: linear in Δt, very very small Δt (larger unstable),
Large errors in Δx → 2nd order BC for ω_0 better?

2nd order BC for ω_0 with $Re = 10$ and $N = 10$, 14 and 20.



Much smaller errors from Δx.

Well matched design

Errors for this problem are 2nd order in Δx and 1st order in Δt,

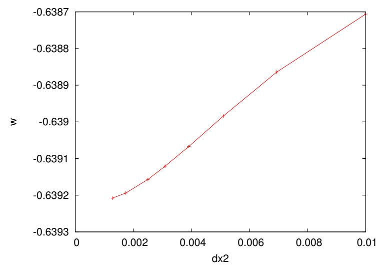
but stability has $\Delta t = \frac{1}{5} Re \Delta x^2$.

Hence time errors $O(\Delta t) \approx$ space errors $O(\Delta x^2)$

Hence no need for second-order time-stepping.

Accuracy consistence. b. Overall $O(\Delta x^2)$

Set $\Delta t = 0.2 Re \Delta x^2$. Plot $\omega(0.5, 0.5, 1)$ at $Re = 10$ for $N = 10$, 12, 14, 16, 18, 20, 24 and 28.

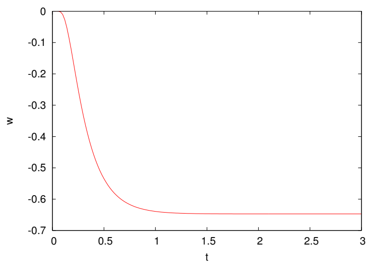


Linear in Δx^2. Result: $\omega(0.5, 0.5, 1) = -0.63925 \pm 0.00005$.

Note linear extrapolation in Δx^2 from $N = 10$ and 14 gives same accuracy as 28 at $\frac{1}{32}$ the CPU.

2.10 Results: time to evolve

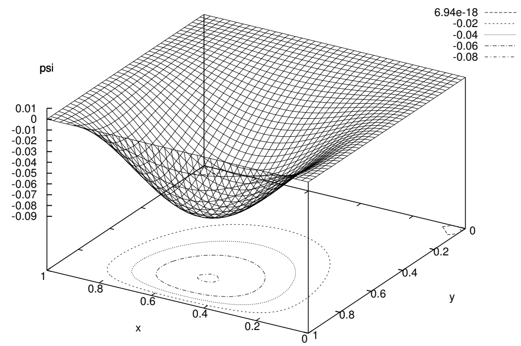
Vorticity at centre of box as a function of time, with $N = 20$ and $Re = 10$.



Steady to 10^{-4} by $t = 2$, time to diffuse across box.
For steady state, try reducing to 3 SOR per time step in place of N.

Results: steady streamfunction

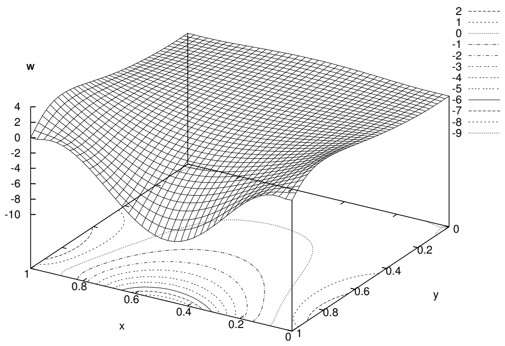
At $t = 3$, $Re = 10$ and $N = 40$.



Fast near lid, slow deep into cavity.
Weak reversed circulations in bottom corners

Results: steady vorticity

At $t = 3$, $Re = 10$ and $N = 40$.

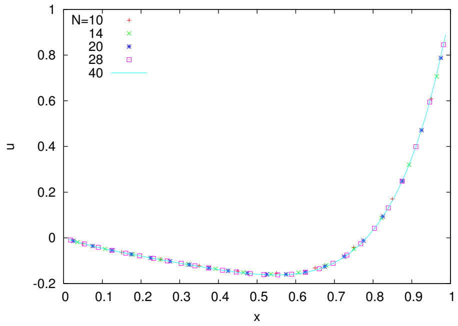


Slight asymmetry downstream

Results: steady mid-section velocity $u(0.5, y)$

$$u_{i\,j+\frac{1}{2}} = \frac{\psi_{i\,j+1} - \psi_{i\,j}}{\Delta x} \quad \text{for} \quad y = (j + \tfrac{1}{2})\Delta x$$

At $Re = 10$, with $N = 10,\ 14,\ 20,\ 28, 40$.



Agree to visual accuracy

Force on lid

$$F = \int_0^1 \left.\frac{\partial u}{\partial y}\right|_{y=1} dx \approx \sum_{i=0}^{N} \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{j=N} \Delta x.$$

With $O(\Delta x)$ error

$$\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{j=N} \approx \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{j=N-1} = \frac{\psi_{i\,N} - 2\psi_{i\,N-1} + \psi_{i,N-2}}{\Delta x^2} + O(\Delta x).$$

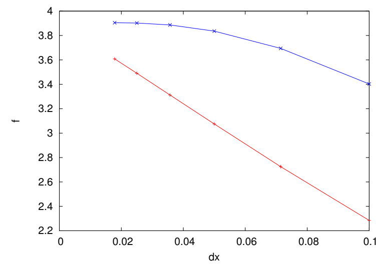
For $O(\Delta x^2)$, linearly extrapolate to boundary

$$\begin{aligned} \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{j=N} &\approx 2\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{j=N-1} - \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{j=N-2} \\ &= \frac{2\psi_{i\,N} - 5\psi_{i\,N-1} + 4\psi_{i,N-2} - \psi_{i,N-3}}{\Delta x^2} + O(\Delta x^2). \end{aligned}$$

Check: $\psi = 1, y, y^2, y^3 \to 0, 0, 2, 0$

Results: force on lid

At $Re = 10$ for $N = 10,\ 14,\ 20,\ 28,\ 40$ and 56.

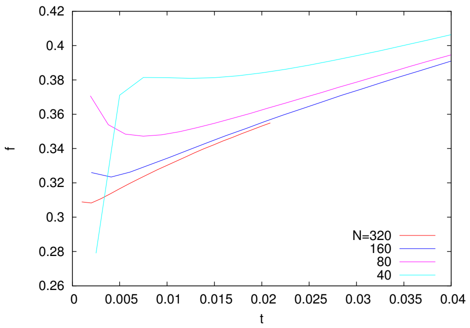


The final answer for the force is

$$F = 3.905 \pm 0.002 \quad \text{at } Re = 10.$$

Results: early times

Simple $\sqrt{\nu t}$ solution. Plot $F/\sqrt{t/Re}$



for $N = 40$, 80, 160 and 320.

Failure: Code not designed for $\sqrt{t}$ behaviour.

Note 0.33, 0.319, 0.307 $\rightarrow \frac{1}{2\sqrt{\pi}} = 0.281$ with $0.4\Delta x^{1/2}$ error.