Example Sheet 1

1. Show that steady simple shear flow $\mathbf{u} = (\gamma y, 0, 0)$ is the sum of a planar extensional flow (whose principal axes should be determined) and a solid body rotation. Show that the Navier-Stokes equations are satisfied if the pressure is constant and the body force vanishes. If the flow is maintained between two plates at $y = 0$ and $y = h$, find the forces on the plates.

2. Consider the two-dimensional linear flow

$$\mathbf{u} = (\alpha x - \tfrac{1}{2}\omega y, -\alpha y + \tfrac{1}{2}\omega x).$$

Confirm that this flow is incompressible and find its streamfunction. Show that the streamlines are elliptic or hyperbolic according to whether $|\alpha| \lessgtr \frac{1}{2}|\omega|$.

Evaluate $\rho \mathbf{u}\cdot\nabla\mathbf{u}$ and find a pressure field to balance it. Discuss the minimal or maximal nature of the pressure at the origin in terms of the streamline pattern.

3. Show for a volume V with a *stationary* rigid boundary that the total rate of dissipation of energy can be written alternatively as

$$2\mu \int_V e_{ij}e_{ij}\, dV = \mu \int_V \omega^2\, dV, \qquad \text{where} \quad \omega = |\nabla \times \mathbf{u}|$$

[It follows that if the flow is irrotational, there is no dissipation: why?]

4. Fluid flows steadily through a cylindrical tube parallel to the z-axis with velocity $\mathbf{u} = (0, 0, w(x, y))$, under a uniform pressure gradient $G = -dp/dz$. Show that the Navier-Stokes equations with no body force are satisfied provided

$$\nabla^2 w = -G/\mu,$$

and state the appropriate boundary conditions.

For a tube with an elliptical cross-section with semi-axes a and b, show that

$$w = w_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

finding w_0. Show that the volume flux (i.e. the volume of fluid passing any section of the tube per unit time) is given by

$$Q = \frac{\pi a^3 b^3 G}{4(a^2 + b^2)\mu}.$$

Now specialise to circular cross-section, $b = a$. Show that the viscous stress on the boundary, which you may take to be $\sigma_{rz} = \mu \partial w/\partial r$, produces an axial force $4\pi\mu w_0 L$ on a length L of the tube, and that this balances the pressure difference exerted across the ends $LG\pi a^2$. Further show that the dissipation within the tube is $2\pi\mu w_0^2 L$ and this is equal to the rate of working against the pressure difference across the ends LGQ.

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5. Two incompressible fluids of the same density ρ and viscosities μ_B and μ_T flow steadily, one on top of the other, down a plane inclined at an angle α to the horizontal. The depths of the layers (normal to the plane) are uniform and equal to h_B and h_T respectively.

Using coordinates x down the plane and y perpendicular to it, write down the boundary conditions on the plane, on the interface between the two layers and on the top free surface. Find the pressure field and velocity field in each fluid on the assumption that they depend only on y. Observe that the velocity profile in the bottom layer depends on h_T but not μ_T. Why?

6. A plane rigid boundary of a semi-infinite domain of fluid oscillates in its own plane with velocity $U \cos \omega t$, and the fluid is at rest at infinity. Find the velocity field. [Hint: use $e^{-\kappa(1+i)z}$ with $\kappa^2 = \omega/2\nu$.] Show that the time-averaged rate of dissipation of energy in the fluid is

$$\tfrac{1}{2}\rho U^2 \left(\tfrac{1}{2}\nu\omega\right)^{1/2}$$

per unit area of the boundary. Verify that this is equal to the time average of the rate of work of the boundary on the fluid (per unit area).

7. Viscous fluid is contained in the space between two coaxial cylinders $r = a$ and $b\ (> a)$, which may be consider to be of infinite length. The inner cylinder rotates with steady angular velocity Ω about its axis and the outer cylinder is at rest. The velocity field in the fluid is steady and of the form $\mathbf{u} = (0, v(r), 0)$ in cylindrical polar coordinates, and the pressure varies only in the radial direction. Look up the components of the Navier-Stokes equations in these coordinates, say in Appendix 2 of Batchelor or Wikipedia. [Alternatively work in Cartesians, with $\mathbf{u} = (yf(r), -xf(r), 0)$ with $r^2 = x^2 + y^2$, using $\partial_x f = xf'/r$.] Show that

$$v(r) = Ar + B/r,$$

where A and B are to be determined. Calculate the torque per unit length that must be applied to the inner cylinder to maintain the motion. [Use the component $e_{\theta r}$ of the strain-rate tensor in cylindrical polars, given by $2e_{\theta r} = r\, d\left(v/r\right)/dr$ in this flow.]

8*. Fluid having kinematic viscosity ν and density ρ is confined between a fixed plate at $y = h$ and a plate at $y = 0$ whose velocity is $(U \cos \omega t, 0, 0)$, where U is a constant. There is no body force and the pressure is independent of x. Explain the physical significance of the dimensionless number $S = \omega h^2/\nu$.

Assuming that the flow remains time-periodic and unidirectional, find expressions for the flow profile and the time-average rate of working Φ per unit area by the plates on the fluid.

Sketch the velocity profile and evaluate Φ in the limits $S \to 0$ and $S \to \infty$, and explain why in these limits Φ becomes independent of ω and h respectively.