Example Sheet 4

1. Wind blowing over a reservoir exerts at the water surface a uniform tangential stress S which is normal to, and away from, a straight side of the reservoir. Use dimensional analysis, based both on balancing the inertial and viscous forces in a thin boundary layer and on the imposed boundary condition, to find order-of-magnitude estimates for the boundary-layer thickness $\delta(x)$ and the surface velocity $U(x)$ as functions of distance x from the shore. Using the boundary-layer equations, find the ordinary differential equation governing the non-dimensional function f defined by

$$\psi(x, y) = U(x)\delta(x)f(\eta), \quad \text{where} \quad \eta = y/\delta(x).$$

What are the boundary conditions on f?

2. A steady two-dimensional jet of fluid runs along a plane rigid wall, the fluid being at rest far from the wall. Use the boundary-layer equations to show that the quantity

$$P = \int_0^\infty u(y) \left(\int_y^\infty u(y')^2 \, dy' \right) dy$$

is independent of the distance x along the wall. Find order-of-magnitude estimates for the boundary-layer thickness and velocity as functions of x.

Show that in the analogous axisymmetric wall jet spreading out radially the velocity varies like $r^{-3/2}$.

3. Show that the streamfunction $\psi(r, \theta)$ for a steady two-dimensional flow satisfies

$$-\frac{1}{r}\frac{\partial(\psi, \nabla^2\psi)}{\partial(r, \theta)} = \nu\nabla^4\psi.$$

Show further that this equation admits solutions of the form

$$\psi = Qf(\theta),$$

if f satisfies

$$f'''' + 4f'' + \frac{2Q}{\nu}f'f'' = 0.$$

[See lectures for solutions.]

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4. A vortex sheet of strength U is located at a distance h above a rigid wall $y = 0$ and is parallel to it, so that the fluid velocity $(u, 0, 0)$ is

$$u = \begin{cases} U & \text{in } 0 < y < h, \\ 0 & \text{in } y > h \ . \end{cases}$$

Suppose now that the sheet is perturbed slightly to the position $y = h + \eta_0 e^{ik(x-ct)}$ where $k > 0$ is real but c may be complex. Show that

$$c = U/(1 \pm i\sqrt{\tanh kh}).$$

Deduce that

(i) the sheet is unstable to disturbances of all wavelengths;

(ii) for short waves ($kh \gg 1$) the growth rate $k\mathrm{Im}(c)$ is $\frac{1}{2}Uk$ and the wave propagation speed $\mathrm{Re}(c)$ is $\frac{1}{2}U$, as if the wall were absent;

(iii) for long waves ($kh \ll 1$) the growth rate is $Uk\sqrt{kh}$ (so that the wall inhibits the growth of long waves) and the propagation speed is U.

5. A two-dimensional jet in the x-direction has velocity profile

$$u = \begin{cases} 0 & \text{in } y > h, \\ U & \text{in } -h < y < h, \\ 0 & \text{in } y < -h \ . \end{cases}$$

The vortex sheets at $y = \pm h$ are perturbed to

$$y = \begin{cases} +h + \eta_1 e^{ik(x-ct)}, \\ -h + \eta_2 e^{ik(x-ct)}. \end{cases}$$

Show that the jet is unstable to a 'varicose' instability for which $\eta_1 = -\eta_2$ (identical to that of question 5), and also to a 'sinuous' instability for which $\eta_1 = \eta_2$ and

$$c = U/(1 \pm i\sqrt{\coth kh}).$$

[The growth rates at small kh are again $Uk\sqrt{kh}$. Hence thin jets (e.g. smoke filaments) can suffer rather slowly growing sinuous instabilities.]

6. Show that the rate of dissipation of mechanical energy in an incompressible fluid is $2\mu e_{ij}e_{ij}$ per unit volume, where e_{ij} is the rate-of-strain tensor and μ is the viscosity.

A finite mass of incompressible fluid, of viscosity μ and density ρ is held in the shape of a sphere $r < a$ by surface tension. It is set into a mode of small oscillations in which the velocity field may be taken to have Cartesian components

$$u = \beta x, \quad v = -\beta y, \quad w = 0.$$

where $\beta \propto \exp(-\epsilon t)\sin\omega t$. Assuming that $\epsilon \ll \omega$, calculate the dissipation rate averaged over a cycle (ignoring the slowly varying factor $\exp(-\epsilon t)$) and hence show that $\epsilon = 5\mu/\rho a^2$. You may assume that the total energy of the oscillation is twice the kinetic energy averaged over a cycle. Why is is permissible to ignore the details of the boundary layer near $r = a$?