Example Sheet 4: Nonlinear Waves

Parts of the old Tripos questions 5 and 9 overlap with earlier questions on this sheet. Where this is the case, you may simply quote the earlier results rather than rederive them.

1. *Shock formation.* At time $t = 0$ the velocity $u(x,t)$ in a one-dimensional simple wave, propagating in the positive x direction through a perfect gas, has the form $u = u_m \sin kx$, where u_m and k are positive constants. Find the time t^* at which shocks form. Sketch $u(x)$ at times $t = 0, t = \frac{1}{2}t^*$ and $t = t^*$. Show that in the time interval $(0, t^*)$ a single *wave-crest* (i.e. a local maximum of $u(x,t)$) travels a distance

$$\frac{1}{k}\left(\frac{2c_0}{(\gamma+1)u_m} + 1\right).$$

Comment: When $k = 2\pi \times (1kHz)/c_0$, $c_0 = 340ms^{-1}$, $\gamma = 1.4$, *and* $u_m = 0.05ms^{-1}$ *(equivalent to 120dB, the pain threshold for the ear), the distance is about 320m.*

2. *Shock formation.* A perfect gas, initially at rest, occupies the region to the right of a piston whose position is $X(t) = \frac{1}{2}at^2$ for $t > 0$. Find the time and position where a shock first forms.

3. *Blood flow.* An artery is modelled as a long straight tube with elastic walls and cross-sectional area $A(x,t)$, which contains incompressible, inviscid blood of density ρ. On the assumption that the fluid velocity u and pressure p do not vary across the artery, conservation of mass and momentum imply that

$$A_t + (uA)_x = 0 \qquad \text{and} \qquad \rho u_t + \rho u u_x = -p_x.$$

The area A is related to the fluid pressure p by an elastic 'tube law' of the form $p = P(A)$, where $P(A)$ is some given, strictly increasing function. Find the Riemann invariants and their corresponding propagation speeds.

Now suppose that

$$P(A) = p_0 + \frac{\rho c_0^2}{2\kappa}\left(\frac{A}{A_0}\right)^{2\kappa},$$

where p_0, A_0, c_0 and κ are positive constants. For $t < 0$ the artery has uniform area A_0 and there is no flow. Blood is then pumped into the artery ($x > 0$) with velocity $U(t)$ at $x = 0$, where

$$U(t) = \begin{cases} U_0 \dfrac{t}{t_1}\Big(2 - \dfrac{t}{t_1}\Big) & (0 \leqslant t \leqslant 2t_1)\ , \\ 0 & (t > 2t_1)\ . \end{cases}$$

Calculate the time and place at which a 'shock' first forms.

Comment: In an adult human, typical values are $A_0 = 5 \times 10^{-4}\,\mathrm{m}^2$, $U_0 = 1.2\,\mathrm{m\,s^{-1}}$, $\kappa = 1$, $c_0 = 5\,\mathrm{m\,s^{-1}}$, $p_0 = 10^4\,\mathrm{N\,m^{-2}}$, $\rho = 10^3\,\mathrm{kg\,m^{-3}}$, $t_1 = 0.35\,\mathrm{s}$. *Do you expect shocks to form?*

4. *A general expansion fan.* A piston confines an inviscid compressible fluid (not necessarily a perfect gas) to the right-hand half, $x > 0$, of an infinite tube. The fluid is initially at rest, $u = 0$, with uniform density ρ_0 and sound speed c_0. For $t > 0$ the piston moves with constant speed V away from the fluid. Assuming that the fluid can keep up with the piston, show that there is a region R_2 in the (x,t)-plane, in which the local sound speed c takes a constant value c_2, which differs from the value c_0 in the undisturbed region R_0. Find an equation that determines c_2 in terms of V and the function $c(\rho)$. Deduce the condition on V for the fluid to keep up with the piston.

Show by *reductio ad absurdum*, or otherwise, that all the C_+ characteristics lying outside both R_2 and R_0 must pass through the origin. Deduce that for $t > 0$

$$u + c = \begin{cases} c_2 - V\ , & -Vt \leqslant x \leqslant (c_2 - V)t \\ xt^{-1}\ , & (c_2 - V)t \leqslant x \leqslant c_0 t \\ c_0\ , & x \geqslant c_0 t \end{cases}$$

Sketch the forms of u and c as functions of x at two different times.

5. *Two expansion fans* (Tripos 75425). A perfect gas, with constant specific heats in the ratio γ, is initially at rest with uniform sound speed c_0. It is confined by two pistons to the region $0 < x < 2\ell$ of a long cylindrical tube. At time $t = 0$, both pistons are set into impulsive motion away from the gas with constant velocities $u = -V < 0$ and $u = U > 0$.

(i) For $0 \leqslant t \leqslant \ell/c_0$ show that in the part $x \leq \ell$ of the tube (which cannot have been reached by any signal from the piston initially at $x = 2\ell$), every C_+ characteristic is a straight line. Show that the fluid velocity u takes the value

$$u = \frac{2}{\gamma+1}\left(\frac{x}{t} - c_0\right) \quad \text{for} \quad \left(c_0 - \frac{\gamma+1}{2}V\right)t < x < c_0 t\ .$$

Give the corresponding value of c. Find the shape of the C_- characteristics when u and c take these values.

(ii) Deduce that, when $t > \ell/c_0$, the equation

$$u = \frac{2}{\gamma+1}\left(\frac{x}{t} - c_0\right)$$

is satisfied only in the smaller interval

$$\left(c_0 - \frac{\gamma+1}{2}V\right)t < x < \frac{\ell}{\gamma-1}\left((\gamma+1)\left(\frac{c_0 t}{\ell}\right)^{(3-\gamma)/(\gamma+1)} - 2\left(\frac{c_0 t}{\ell}\right)\right)\ .$$

(iii) For a case with V/c_0 about $\frac{1}{2}$ and U/c_0 about $\frac{1}{4}$, give a rough sketch indicating **four** areas of the (x,t) plane throughout each of which u takes a different constant value, to be specified.

6.* *Expansion fan and escape velocity.* Consider the situation in question 4 for the case of a perfect gas with specific-heat ratio γ. Find the equations in regions R_0, R_1 and R_2 of

(i) the C_- characteristic that originates at $x = \xi$ and $t = 0$

(ii) the trajectory of the gas particle which is at $x = \xi$ when $t = 0$.

Sketch the C_+ and C_- characteristics and the particle trajectories in the (x,t)-plane. Hence explain what happens when $V > 2(\gamma-1)^{-1}c_0$.

7. *A piston-generated shock.* A piston moves with constant positive velocity u_1 into a perfect gas of specific heat ratio $\gamma > 1$, generating a shock wave which moves ahead of the piston. Show that a possible solution of all the relevant equations is one in which the gas is at rest beyond the shock, at pressure p_0, and is moving with constant velocity u_1 in the region between the piston and the shock, throughout which region the density and pressure also take constant values ρ_1, p_1 which are determined by

$$\frac{\rho_1}{\rho_0} = \frac{2\gamma + (\gamma+1)\beta}{2\gamma + (\gamma-1)\beta}\ , \qquad \frac{1}{\beta^2} + \frac{\gamma+1}{2\gamma\beta} = \frac{c_0^2}{\gamma^2 u_1^2}\ ,$$

where β is the shock strength defined as $(p_1 - p_0)/p_0 > 0$, and ρ_0 and c_0 are the density and sound speed of the undisturbed gas. Show also that the shock speed $V = c_0(1 + \frac{\gamma+1}{2\gamma}\beta)^{1/2}$.

8. *Traffic flow.* Assume that the speed of cars down a long straight (one-way) road is a known, monotonically decreasing function $u(\rho)$ of the local density ρ of traffic. The flux of cars is thus given by $q(\rho) = \rho u$. From conservation of cars deduce that ρ is constant on characteristics $dx/dt = c(\rho)$, where $c = dq/d\rho$. Deduce also that if a shock develops between regions of density ρ_1 and ρ_2 then it propagates with speed $[q(\rho_1) - q(\rho_2)]/(\rho_1 - \rho_2)$.

Consider the case $u(\rho) = U(1 - \rho/\rho_0)$ where U is (10% faster than) the speed limit and ρ_0 is the density of a nose-to-tail traffic jam. Sketch the functions $q(\rho)$ and $c(\rho)$. Explain why shocks only form when light traffic is behind heavy traffic, and why the shocks can travel either forwards or backwards depending on the density of traffic.

A queue of cars with density ρ_0 is waiting in $-L < x < 0$ behind a red traffic light at $x = 0$. There are no other cars on the road. The light turns green at $t = 0$. Find the time T when the last car starts to move, and determine the velocity of the last car for $t > T$. [*Hint:* The solution involves both a shock and an expansion fan.]

9.* *A method to generate shock waves in a 'shock tube'* (Tripos 85427). An infinitely long uniform tube contains two perfect gases separated by a membrane at $x = 0$. The gas in $x > 0$ has pressure p_1, density ρ_1 and specific heat ratio γ_1; the corresponding values for the gas in $x < 0$ are p_2, ρ_2, γ_2 where $p_2 > p_1$. At $t = 0$ the membrane is burst. Assuming that the interface between the two gases remains plane and moves with constant speed V, use the one-dimensional equations of motion to show that there are three regions in the tube in which the pressure is uniform,

$$
\begin{aligned}
p = p_2 &\quad \text{for} \quad x < -\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2} t \,, \\
p = p_1 &\quad \text{for} \quad x > Ut \,, \\
p = p_m &\quad \text{for} \quad -\left[\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2} - \frac{\gamma_2 + 1}{2} V\right] t < x < Ut \,,
\end{aligned}
$$

where p_m is as yet unknown, and the shock velocity, U, is a constant to be found in terms of p_m, p_1, ρ_1, γ_1.

Show that V is related to p_m by the following two equations:

$$
\begin{aligned}
V &= (p_m - p_1)\left(\tfrac{1}{2}\rho_1\left[(\gamma_1 + 1)p_m + (\gamma_1 - 1)p_1\right]\right)^{-1/2} \,, \\
V &= \frac{2}{\gamma_2 - 1}\left(\frac{\gamma_2 p_2}{\rho_2}\right)^{1/2}\left[1 - \left(\frac{p_m}{p_2}\right)^{(\gamma_2 - 1)/2\gamma_2}\right] \,,
\end{aligned}
$$

and hence show that there is a unique solution for p_m and V.