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Example Sheet 2

1. *SHO.* The needle of a Ballistic Galvanometer is a critically damped simple harmonic oscillator, i.e. $\gamma = \Omega$ in $\ddot{x} + 2\gamma\dot{x} + \Omega^2 x = 0$. On receipt of a small electric charge the needle experiences an impulse, i.e. $\dot{x} = V$ and $x = 0$ at $t = 0$. Find the maximum displacement and the time for the displacement to decay to 1% of the maximum. (A solution to $xe^{-x} = 1/100e$ is $x = 7.63$.)

Setting $\gamma = \Omega \cosh\alpha$, obtain an expression for $\dot{x}$ involving $\sinh(\alpha - \Omega t \sinh\alpha)$. Hence show that, as γ increases for fixed Ω, the time to the maximum displacement and the maximum displacement decrease. [* The time to decay to 1% increases. *]

2. *Seismograph.* A mass m is suspended below a table by a spring with constant k and in parallel by a dashpot with constant c. The distance between the table and the mass, $x(t)$, is recorded. The table is shaken vertically (relative to a inertial frame) according to $y(t) = \Re\{y_0 e^{i\omega t}\}$. Find an expression for the amplitude of forced response in x. In the special case of $c = \sqrt{2km}$, show that this recorded amplitude of x is within 5% of y_0 if $\omega > 1.8\sqrt{k/m}$.

3. *Dry friction.* Consider a mass m at position $x(t)$ on a rough horizontal table attached to the origin by a spring with constant k and with a dry friction force f

$$\begin{cases} f = F & \text{if } \dot{x} < 0 \\ -F \le f \le F & \text{if } \dot{x} = 0 \\ f = -F & \text{if } \dot{x} > 0 \end{cases} .$$

What is the range of x where the mass can rest? Show that if the mass moves, the maximum excursion decreases by $2F/k$ per half cycle. Discuss the motion.

4. *Quadratic friction.* A particle is projected vertically upwards at a speed v_0 and returns to the same position with a speed v_1. The quadratic friction (proportional to v^2, and opposing the upward *and* downward velocity) produces a terminal velocity v_∞. Show that

$$\frac{1}{v_1^2} = \frac{1}{v_0^2} + \frac{1}{v_\infty^2} .$$

(Note that the return velocity is always less than the terminal velocity.) Show that the time taken is

$$\frac{v_\infty}{g}\left[\tan^{-1}(v_0/v_\infty) + \tanh^{-1}(v_1/v_\infty)\right] .$$

5. *Varying mass.* A rain drop of mass $m(t)$ falls at velocity $v(t)$ under gravity with negligible air resistance through a cloud of much smaller water droplets. Take the rate of accretion as

$$\dot{m} = \rho_v \pi a^2 v .$$

Here $a(t)$ is the radius of the drop, i.e. $m = \rho_l \frac{4}{3}\pi a^3$, and $\rho_{l,v}$ are the densities of water in the drop and in the cloud. Taking $a(0) = v(0) = 0$, find how the drop accelerates and grows, by first using dimensional analysis to find the form of the expressions and then substituting into the governing equations to fix the unknown coefficients.

Please notify all errors to `E.J.Hinch@damtp.cam.ac.uk`.

6. *Rocket.* A two stage rocket consists of (i) a second stage of total mass m, payload p and fuel $(m-p)\theta$, and (ii) a first stage of total mass M, payload m and fuel $(M-m)\theta$, where $0 < \theta < 1$. Mass is ejected from the two stages at the constant relative velocity c. Ignoring gravity and air resistance, show that starting from rest the final velocity of the payload p is

$$c \ln \frac{M}{m\theta + M(1-\theta)} + c \ln \frac{m}{p\theta + m(1-\theta)} .$$

Show that this is greater than a single stage rocket with the same total mass M and payload p. (Why?) Show that, if the total mass M, payload p and fraction of fuel θ are given, the optimal size of the second stage is $m = \sqrt{Mp}$. (Guess the generalization to further stages.) If $c = 2.5\,\mathrm{km\,s^{-1}}$ and $\theta = 0.9$ (and $p \ll M$), show that a two stage rocket can attain the escape velocity of $11.2\,\mathrm{km\,s^{-1}}$ whereas a single stage cannot.

7. *Energy.* The Great Pyramid at Gizeh is 150 m high, has a 230 m long square base, and is effectively constructed of solid stone with a density $2500\,\mathrm{kg\,m^{-3}}$. Find the total gravitational potential energy in raising the stone above ground level.

If slaves consume food of $1500\,\mathrm{Cal\,day^{-1}}$, 10% of which is available for useful work, how many slaves would be required to do the work over 20 years? (Herodotus recorded that 100,000 were required for the full construction.)

8. *Power, V easy.* A car of mass 1 tonne moves at that constant acceleration which takes it from rest to 60 mph in 10 seconds. What power (in kW) is required at 50 mph?

9. *Small amplitude oscillations.* The mutual potential energy of a Li^+ ion and an I^- ion as a function of their separation r may be approximated by

$$V(r) = -\frac{e^2}{4\pi\epsilon r} + \frac{A}{r^{10}} ,$$

where $1/4\pi\epsilon = 8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$ and $e = 1.6 \times 10^{-19}\,\mathrm{C}$. The equilibrium distance r_0 between the centres of the ions in the LiI molecule is about 2.4 Å (1 Å $= 10^{-10}$ m). Find the constant A, and thence the work (in eV) required to tear the ions appart. Taking the I^- ion to be fixed (because it is so massive) and the mass of the Li^+ ion to be 10^{-26} kg, what is the vibrational frequency of the molecule?

10. *Energy conservation.* Consider a particle moving on a smooth table attached by a spring of natural length l to a point l above the table. Show that, for displacements x from the equilibrium position small compared with l, the restoring force varies as $-Cx^3$. Using dimensional analysis (only with m, C and x_{max}) find how the period of this nonlinear oscillator varies with the amplitude (maximum displacement). Using energy conservation and $\int_0^1 (1-\xi^4)^{-1/2} = 1.311$ to find the constant of proportionality.

11. *Escape velocities.* What is the escape velocity from the Earth, the Moon (mass $M_E/81$, radius $0.27R_E$) and Mars (mass $0.11M_E$, radius $0.53R_E$)? The average thermal velocity $\sqrt{3kT/m}$ of H_2 (mass $m = 2$ amu) in the Earth's atmosphere is $1770\,\mathrm{m\,s^{-1}}$. Find the thermal velocities of CO_2 (44 amu), N_2 (28 amu), NH_3 (17 amu) and H_2 on the Earth, and on the Moon and Mars supposing that their atmposheres would have temperatures of T_E and $0.8T_E$ respectively. Over 10^9 years, a gas would escape if its average thermal velocity exceeded 20% of the escape velocity. Which gases would certainly be lost by which atmospheres?