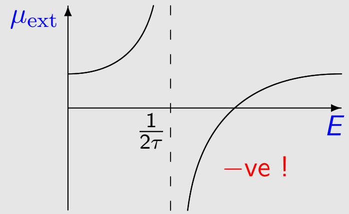
Chapter 10 – Strong flows

- Birefreingent strand
  - – thin layer of high stress leaqving a stagnation point
- Wine-glass model of contraction flow
  - – anisotropic flow from anisotropic material
- Corner singularity
  - – fast flow with no relaxation
- Limited-forec flows
  - – strain only to avoid relaxation

Oldroyd-B, and its limitations

$$\frac{DA}{Dt} = A \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot A - \frac{1}{\tau}(A - \mathbf{I})$$
$$\sigma = -p\mathbf{I} + 2\mu_0 E + G f A$$

Steady extensional flow



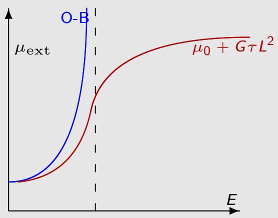
Microstructure deforms without limit if $E > \frac{1}{2\tau}$: $A = e^{(2E - \frac{1}{\tau})t}$

Need to limit deformation of microstructure

FENE modification

Finite Extension Nonlinear Elasticity

$$\frac{DA}{Dt} = A \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot A - \frac{f}{\tau}(A - \mathbf{I})$$
$$\sigma = -p\mathbf{I} + 2\mu_0 E + G f A$$
$$f = \frac{L^2}{L^2 - \operatorname{trace} A} \quad \text{keeps} \quad A < L^2$$



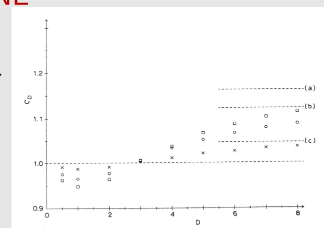
Will use FENE, and if safe Oldroyd-B, in following strong flows

FENE flow past a sphere

Oldroyd-B gave decrease is drag

FENE

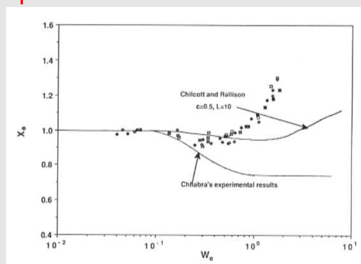
F



De

Chilcott & Rallison 1988 JNNFM

Experiments M1



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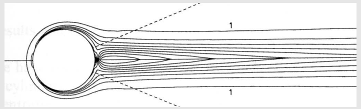
Tirtaatmadja, Uhlherr & Sridhar 1990 JNNFM

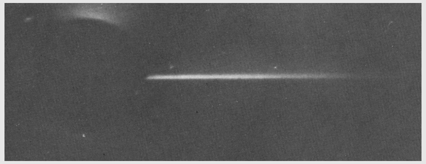
FENE gives drag increase

. . . FENE flow past sphere

FENE drag increase from long wake of high stress



Chilcott & Rallison 1988 JNNFM

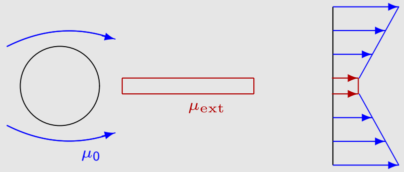


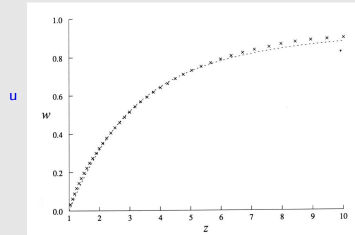
Cressely & Hocquart 1980 Opt Act

"Birefringent strand"

. . . birefringent strands

Boundary layers of high stress.
Crude model: μ_{ext} in wake, μ_0 elsewhere.

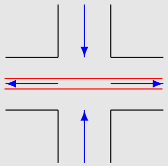


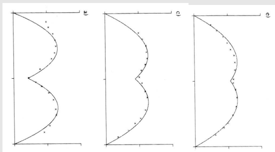


Harlen, Rallison & Chilcott 1990 JNNFM

. . . birefringent strands

Can apply to all flows with stagnation points, e.g.

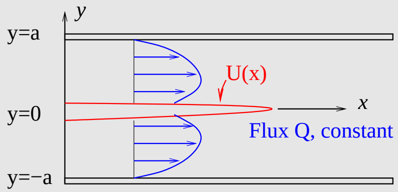




Harlen, Rallison & Chilcott 1990 JNNFM

Also cusps at rear stagnation point of bubbles.

Analysis of birefringent strand in exit channel



Lubrication flow

$$u(x,y) = U(x)\frac{a-y}{a} + (Q - Ua)\frac{3y(a-y)}{a^2}$$

Force balance on strand

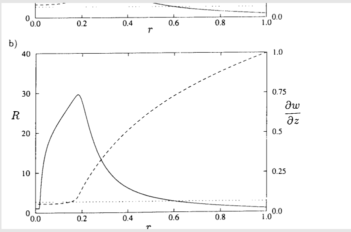
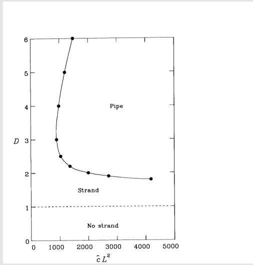
$$\left[\mu\frac{\partial u}{\partial y}\right]_{0-}^{0+} + \frac{\partial}{\partial x}\left(\delta\mu_{\text{ext}}\frac{\partial U}{\partial x}\right)$$

Solving (Student Exercise)

$$U(x) = \frac{3Q}{2a}\left(1 - e^{-\sqrt{\frac{8\mu}{\delta\mu_{\text{ext}}a}}x}\right)$$

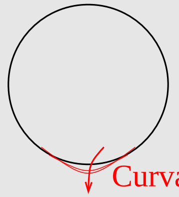
Birefringent pipes

Very low extension rate in the strand can fail to stretch the microstruture, so relax, producing birefringent “pipes”.

Harlen, H, Rallison (1992) JNNFM 44

Formation of a cusp at rear stagnation point of a bubble



Curvature increases with extensibility of fluid

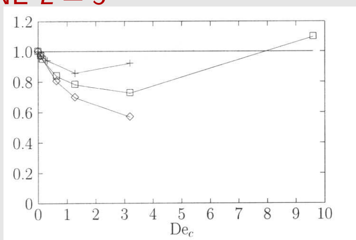
Rallsion & Malaga (2007) JNNFM 141

FENE contraction flow

Oldroyd-B gave decrease is pressure drop

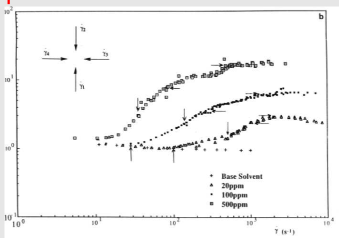
FENE $L = 5$

Δp



Szabo, Rallison & Hinch 1997 JNNFM

Experiments



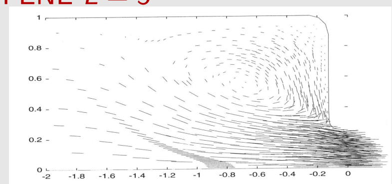
Cartalos & Piau 1992 JNNFM

FENE gives increase in pressure drop

. . . FENE contraction flow

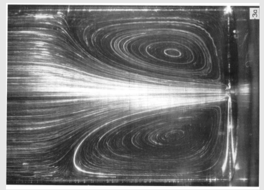
Increase in pressure drop from long upstream vortex

FENE $L = 5$



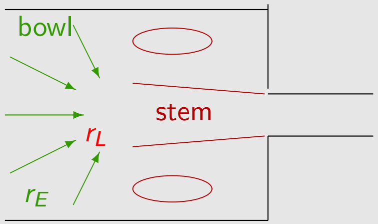
Szabo, Rallison & Hinch 1997 JNNFM

Experiments



Cartalos & Piau 1992 JNNFM

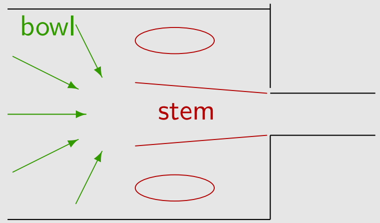
. . . a champagne-glass model



Bowl:

- Sink flow $\quad u = \frac{Q}{2\pi r^2}$
- Stretching starts at $\frac{1}{\tau} = E = \frac{\partial u}{\partial r}$, $\quad$ i.e. at $r_E = (Q\tau)^{1/3}$
- Then deforms as $A \propto u^2 \propto r^{-4}$
- So fully stretched at $A \approx L^2$, $\quad$ at $r_L = r_E/L^{1/2}$
- Hence fully stretched only if $De = \frac{Q\tau}{d^3} > L^{3/2}$.

. . . a champagne-glass model



Stem:

- Fully stretched, $A \approx L^2$, so $\mu_{\text{ext}} = \mu_0 + G\tau L^2 \gg \mu_0 = \mu_{\text{shear}}$
- Balance $\quad \mu_{\text{ext}} \dfrac{\partial^2 u}{\partial r^2} = \mu_{\text{shear}} \dfrac{1}{r^2} \dfrac{\partial^2 u}{\partial \theta^2}$
- By small cone angle $\Delta\theta = \sqrt{\frac{\mu_{\text{shear}}}{\mu_{\text{ext}}}}$
- Length of cone $(r_L - r_c)/\Delta\theta$.
- Start up possible.

Flow anisotropy from material anisotropy: $\mu_{\text{ext}} \gg \mu_{\text{shear}}$ TDR

Fast flows with no relaxation

If $\nabla \mathbf{u} \gg \frac{1}{\tau}$

$$\frac{DA}{Dt} = A \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot A - \frac{1}{\tau}(A - \mathbf{I})$$

Recall material line elements

$$\frac{d}{dt}\delta\ell = \delta\ell \cdot \nabla \mathbf{u},$$

So $\delta\ell$ stretches when $\mathbf{u}$ increases, in steady flow $\delta\ell \propto \mathbf{u}$

Suggests steady solution $\quad$ ($g(\psi)$ from matching to slower region)

$$A = g(\psi)\mathbf{uu}, \quad \text{so } \sigma = -p\mathbf{I} + 2\mu_0 E + Gg\mathbf{uu}$$

Tensions in streamlines again

Fast flows with no relaxation 2

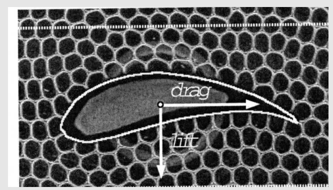
Momemtum, ignoring viscous stress

$$0 = -\nabla p + Gg^{1/2}\mathbf{u} \cdot \nabla g^{1/2}\mathbf{u}.$$

Euler equation!!

Anti-Bernoulli

$$p - \tfrac{1}{2}Ggu^2 = \text{const}$$



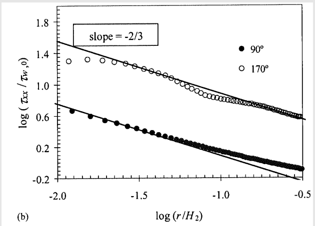
Dollet, Aubouy & Graner 2005 PRL

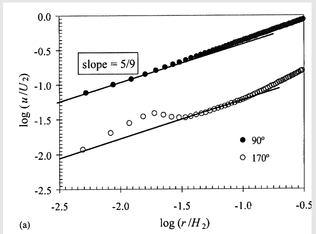
Fast flows with no relaxation 3

Potential flows $g^{1/2}\mathbf{u} = \nabla\phi$

Flow around sharp 270° corner: Hinch 1995 JNNFM

$$\phi = r^{2/3}\cos\tfrac{2}{3}\theta, \qquad \sigma \propto r^{-2/3} \qquad \psi = r^{14/9}\sin^{7/3}\tfrac{2}{3}\theta$$





Alves, Oliviera & Pinho 2003 JNNFM

Fast flows with no relaxation 4

The matching for ψ

$$g^{1/2}(\psi)\nabla\times(0,0,\psi) = g^{1/2}\mathbf{u} = \nabla\phi = \nabla\times(0,0,\tfrac{3}{2}r^{2/3}\sin\tfrac{2}{3}\theta)$$

so

$$\psi = f(r^{3/2}\sin\tfrac{2}{3}\theta) \sim f(r^{2/3}\theta) \quad \text{at small } \theta.$$

$$\begin{cases}\text{In fast core, } De \geq 1 & A_{rr} = gu^2 = r^{-2/3}\\ \text{Near bndry, } De \leq 1 & A_{rr} = 1 + 2\gamma^2\end{cases} \qquad \text{Match: } \begin{cases}\gamma = r^{-1/3}\\ 1 = De = \frac{u}{r}\end{cases}$$

Now near the boundary

$$r = u = \gamma r\theta, \quad \text{so } \theta = r^{1/3}, \quad \text{so } \psi = \gamma(r\theta)^2 = r^{7/3} = (r^{2/3}\theta)^{7/3}$$

Hence elsewhere

$$\psi = Cr^{14/9}\sin^{7/3}\tfrac{2}{3}\theta.$$

Details of the boundary layers – very difficult

Deforming with the flow

While line elements parallel to the flow are stretched $\propto u$, perpendicular elements are squashed $\propto 1/u$, plus some shear. Hence try

$$A = \lambda\mathbf{uu} + \mu(\mathbf{uv} + \mathbf{vu}) + \nu\mathbf{vv}$$

with

$$\mathbf{u}\cdot\mathbf{v} = 0, \; v = 1/u$$

Oldroyd-B becomes Student Exercise

$$\mathbf{u}\cdot\nabla\lambda = \frac{2\gamma}{u^2}\mu - \frac{1}{\tau}\left(\lambda - \frac{1}{u^2}\right)$$

$$\mathbf{u}\cdot\nabla\mu = \frac{\gamma}{u^2}\nu \qquad -\frac{1}{\tau}\mu$$

$$\mathbf{u}\cdot\nabla\nu = \qquad -\frac{1}{\tau}(\nu - u^2)$$

with

$$\gamma = \mathbf{v}\cdot(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\cdot\mathbf{u} = -u^2\nabla\cdot\mathbf{v}$$

Renardy (1994) JNNFM 52

Capillary squeezing – controlled by relaxation

Mass $\dot{a} = -\frac{1}{2}Ea$

Momentum $\frac{\chi}{a} = 3\mu_0 E + G(A_{zz} - A_{rr})$

Microstructure $\dot{A}_{zz} = 2EA_{zz} - \frac{1}{\tau}(A_{zz} - 1)$

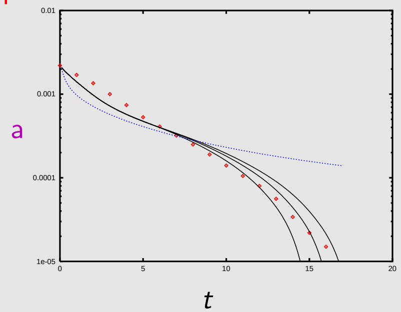
Solution $a(t) = a(0)e^{-t/3\tau}$ Student Exercise

Need slow $E = 1/3\tau$ to stop A_{zz} relaxing from χ/Ga

. . . capillary squeezing

Oldroyd-B $\quad a(t) = a(0)e^{-t/3\tau} \quad$ does not break

Experiments S1 fluid



Exp: Liang & Mackley 1994 JNNFM

Thy: Entov & Hinch 1997 JNNFM

but filament eventually breaks in experiments

Multi-mode generalisation

$$\dot{A}^i_{zz} = 2\left(E = -2\frac{\dot{a}}{a}\right) A^i_{zz} - \frac{1}{\tau_i} A^i_{zz}$$

So

$$A^i_{zz} = \frac{1}{a^4(t)} e^{-t/\tau_i}$$

Hence momentum equation

$$\frac{\chi}{a} = \frac{1}{a^4} \sum g_i e^{-t/\tau_i}$$

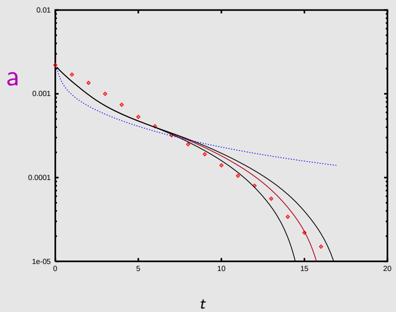
i.e.

$$a(t) = \left(\frac{G(t)}{\chi}\right)^{1/3} \quad \text{with relaxation} \quad G(t) = \sum g_i e^{-t/\tau_i}$$

Spectrum needed to fit experiments at middle times

FENE capillary squeezing

Filament breaks in with FENE $L = 20$



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