

Chapter 10 – Strong flows

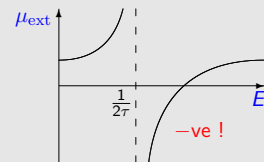
- ▶ Birefringent strand
 - thin layer of high stress leaving a stagnation point
- ▶ Wine-glass model of contraction flow
 - anisotropic flow from anisotropic material
- ▶ Corner singularity
 - fast flow with no relaxation
- ▶ Limited-forec flows
 - strain only to avoid relaxation

Oldroyd-B, and its limitations

$$\frac{DA}{Dt} = A \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot A - \frac{1}{\tau} (A - \mathbf{I})$$

$$\sigma = -p\mathbf{I} + 2\mu_0 E + GfA$$

Steady extensional flow



Microstructure deforms without limit if $E > \frac{1}{2\tau}$; $A = e^{(2E - \frac{1}{\tau})t}$

Need to limit deformation of microstructure

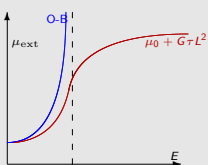
FENE modification

Finite Extension Nonlinear Elasticity

$$\frac{DA}{Dt} = A \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot A - \frac{f}{\tau} (A - \mathbf{I})$$

$$\sigma = -p\mathbf{I} + 2\mu_0 E + GfA$$

$$f = \frac{L^2}{L^2 - \text{trace } A} \quad \text{keeps } A < L^2$$

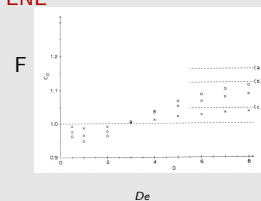


Will use FENE, and if safe Oldroyd-B, in following strong flows

FENE flow past a sphere

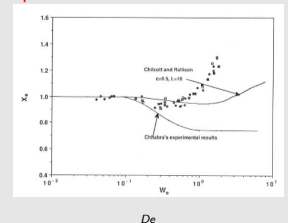
Oldroyd-B gave decrease in drag

FENE



Chilcott & Rallison 1988 JNNFM

Experiments M1

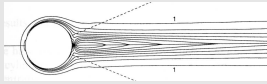


Tirataatmadja, Uhlherr & Sridhar 1990 JNNFM

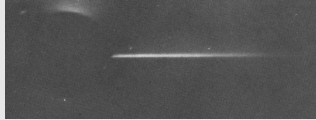
FENE gives drag increase

... FENE flow past sphere

FENE drag increase from long wake of high stress



Chilcott & Rallison 1988 JNNFM



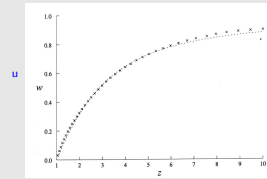
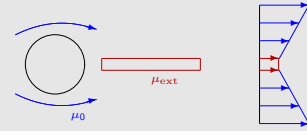
Cressely & Hocquart 1980 Opt Act

"Birefringent strand"

... birefringent strands

Boundary layers of high stress.

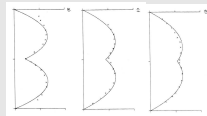
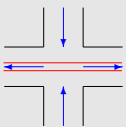
Crude model: μ_{ext} in wake, μ_0 elsewhere.



Harlen, Rallison & Chilcott 1990 JNNFM

... birefringent strands

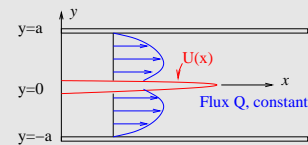
Can apply to all flows with stagnation points, e.g.



Harlen, Rallison & Chilcott 1990 JNNFM

Also cusps at rear stagnation point of bubbles.

Analysis of birefringent strand in exit channel



Lubrication flow

$$u(x, y) = U(x) \frac{a-y}{a} + (Q - Ua) \frac{3y(a-y)}{a^2}$$

Force balance on strand

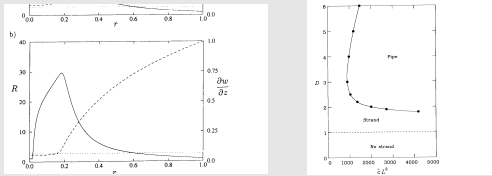
$$\left[\mu \frac{\partial u}{\partial y} \right]_{0-}^{0+} + \frac{\partial}{\partial x} \left(\delta \mu_{ext} \frac{\partial U}{\partial x} \right)$$

Solving (Student Exercise)

$$U(x) = \frac{3Q}{2a} \left(1 - e^{-\sqrt{\frac{\delta \mu}{\delta \mu_{ext} a^2}} x} \right)$$

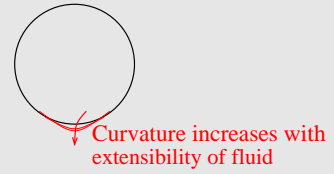
Birefringent pipes

Very low extension rate in the strand can fail to stretch the microstructure, so relax, producing birefringent "pipes".



Harlen, H, Rallison (1992) JNNFM 44

Formation of a cusp at rear stagnation point of a bubble

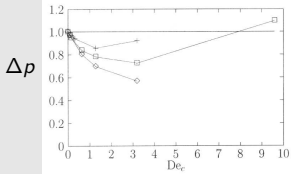


Rallison & Malaga (2007) JNNFM 141

FENE contraction flow

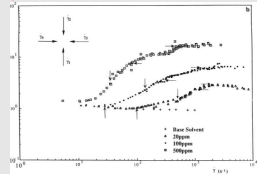
Oldroyd-B gave decrease in pressure drop

FENE $L = 5$



Szabo, Rallison & Hinch 1997 JNNFM

Experiments



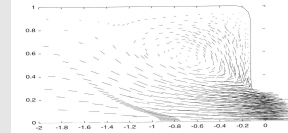
Cartalos & Piau 1992 JNNFM

FENE gives increase in pressure drop

... FENE contraction flow

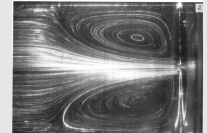
Increase in pressure drop from long upstream vortex

FENE $L = 5$



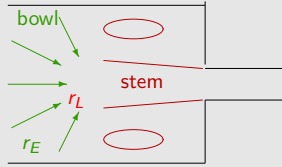
Szabo, Rallison & Hinch 1997 JNNFM

Experiments



Cartalos & Piau 1992 JNNFM

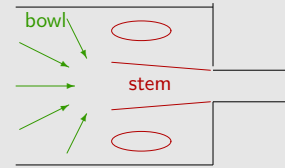
... a champagne-glass model



Bowl:

- ▶ Sink flow $u = \frac{Q}{2\pi r^2}$
- ▶ Stretching starts at $\frac{1}{\tau} = E = \frac{\partial u}{\partial r}$, i.e. at $r_E = (Q\tau)^{1/3}$
- ▶ Then deforms as $A \propto u^2 \propto r^{-4}$
- ▶ So fully stretched at $A \approx L^2$, at $r_L = r_E/L^{1/2}$
- ▶ Hence fully stretched only if $De = \frac{Q\tau}{\delta^3} > L^{3/2}$.

... a champagne-glass model



Stem:

- ▶ Fully stretched, $A \approx L^2$, so $\mu_{\text{ext}} = \mu_0 + G\tau L^2 \gg \mu_0 = \mu_{\text{shear}}$
- ▶ Balance $\mu_{\text{ext}} \frac{\partial^2 u}{\partial r^2} = \mu_{\text{shear}} \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$
- ▶ By small cone angle $\Delta\theta = \sqrt{\frac{\mu_{\text{shear}}}{\mu_{\text{ext}}}}$
- ▶ Length of cone $(r_L - r_c)/\Delta\theta$.
- ▶ Start up possible.

Flow anisotropy from material anisotropy: $\mu_{\text{ext}} \gg \mu_{\text{shear}}$ TDR

Fast flows with no relaxation

If $\nabla \mathbf{u} \gg \frac{1}{\tau}$

$$\frac{DA}{Dt} = A \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot A - \frac{1}{\tau} (A - \mathbf{I})$$

Recall material line elements

$$\frac{d}{dt} \delta \ell = \delta \ell \cdot \nabla \mathbf{u},$$

So $\delta \ell$ stretches when \mathbf{u} increases, in steady flow $\delta \ell \propto \mathbf{u}$

Suggests steady solution ($g(\psi)$ from matching to slower region)

$$A = g(\psi) \mathbf{u} \mathbf{u}, \quad \text{so } \sigma = -p \mathbf{I} + 2\mu_0 E + Gg \mathbf{u} \mathbf{u}$$

Tensions in streamlines again

Fast flows with no relaxation 2

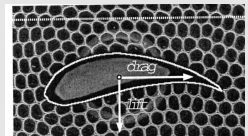
Momentum, ignoring viscous stress

$$0 = -\nabla p + Gg^{1/2} \mathbf{u} \cdot \nabla g^{1/2} \mathbf{u}.$$

Euler equation!!

Anti-Bernoulli

$$p - \frac{1}{2} Gg u^2 = \text{const}$$



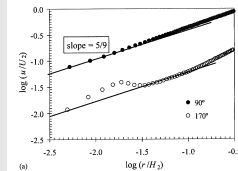
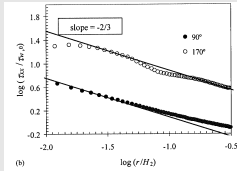
Dollet, Aubouy & Graner 2005 PRL

Fast flows with no relaxation 3

Potential flows $g^{1/2}\mathbf{u} = \nabla\phi$

Flow around sharp 270° corner: Hinch 1995 JNNFM

$$\phi = r^{2/3} \cos \frac{2}{3}\theta, \quad \sigma \propto r^{-2/3} \quad \psi = r^{14/9} \sin^{7/3} \frac{2}{3}\theta$$



Alves, Oliveira & Pinho 2003 JNNFM

Fast flows with no relaxation 4

The matching for ψ

$$g^{1/2}(\psi)\nabla \times (0, 0, \psi) = g^{1/2}\mathbf{u} = \nabla\phi = \nabla \times (0, 0, \frac{3}{2}r^{2/3} \sin \frac{2}{3}\theta)$$

so

$$\psi = f(r^{3/2} \sin \frac{2}{3}\theta) \sim f(r^{2/3}\theta) \quad \text{at small } \theta.$$

$$\begin{cases} \text{In fast core, } De \geq 1 & A_{rr} = gu^2 = r^{-2/3} \\ \text{Near bndry, } De \leq 1 & A_{rr} = 1 + 2\gamma^2 \end{cases} \quad \text{Match: } \begin{cases} \gamma = r^{-1/3} \\ 1 = De = \frac{u}{r} \end{cases}$$

Now near the boundary

$$r = u = \gamma r\theta, \quad \text{so } \theta = r^{1/3}, \quad \text{so } \psi = \gamma(r\theta)^2 = r^{7/3} = (r^{2/3}\theta)^{7/3}$$

Hence elsewhere

$$\psi = Cr^{14/9} \sin^{7/3} \frac{2}{3}\theta.$$

Details of the boundary layers – very difficult

Deforming with the flow

While line elements parallel to the flow are stretched $\propto u$, perpendicular elements are squashed $\propto 1/u$, plus some shear. Hence try

$$A = \lambda\mathbf{u}\mathbf{u} + \mu(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}) + \nu\mathbf{v}\mathbf{v}$$

with

$$\mathbf{u} \cdot \mathbf{v} = 0, \quad v = 1/u$$

Oldroyd-B becomes **Student Exercise**

$$\mathbf{u} \cdot \nabla \lambda = \frac{2\gamma}{u^2} \mu - \frac{1}{\tau} \left(\lambda - \frac{1}{u^2} \right)$$

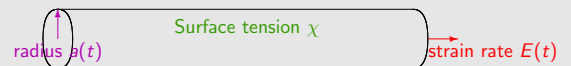
$$\mathbf{u} \cdot \nabla \mu = \frac{\gamma}{u^2} \nu - \frac{1}{\tau} \mu$$

$$\mathbf{u} \cdot \nabla \nu = -\frac{1}{\tau} (\nu - u^2)$$

with

$$\gamma = \mathbf{v} \cdot (\nabla\mathbf{u} + \nabla\mathbf{u}^T) \cdot \mathbf{u} = -u^2 \nabla \cdot \mathbf{v}$$

Capillary squeezing – controlled by relaxation



$$\text{Mass} \quad \dot{a} = -\frac{1}{2} E a$$

$$\text{Momentum} \quad \frac{\chi}{a} = 3\mu_0 E + G(A_{zz} - A_{rr})$$

$$\text{Microstructure} \quad \dot{A}_{zz} = 2EA_{zz} - \frac{1}{\tau}(A_{zz} - 1)$$

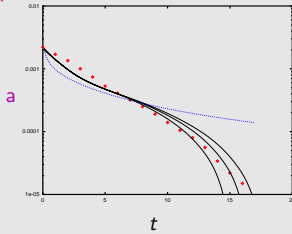
$$\text{Solution} \quad a(t) = a(0)e^{-t/3\tau} \quad \text{Student Exercise}$$

Need slow $E = 1/3\tau$ to stop A_{zz} relaxing from χ/Ga

... capillary squeezing

Oldroyd-B $a(t) = a(0)e^{-t/3\tau}$ does not break

Experiments S1 fluid



Exp: Liang & Mackley 1994 JNNFM
Thy: Entov & Hinch 1997 JNNFM

but filament eventually breaks in experiments

Multi-mode generalisation

$$\dot{A}_{zz}^i = 2 \left(E = -2 \frac{\dot{a}}{a} \right) A_{zz}^i - \frac{1}{\tau_i} A_{zz}^i$$

So

$$A_{zz}^i = \frac{1}{a^4(t)} e^{-t/\tau_i}$$

Hence momentum equation

$$\frac{\chi}{a} = \frac{1}{a^4} \sum g_i e^{-t/\tau_i}$$

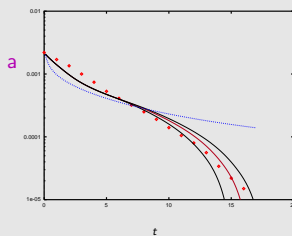
i.e.

$$a(t) = \left(\frac{G(t)}{\chi} \right)^{1/3} \quad \text{with relaxation} \quad G(t) = \sum g_i e^{-t/\tau_i}$$

Spectrum needed to fit experiments at middle times

FENE capillary squeezing

Filament breaks in with FENE $L = 20$



Exp: Liang & Mackley 1994 JNNFM
Thy: Entov & Hinch 1997 JNNFM