No lecture Thursday 17 February 2011

Next lecture Tuesday 22 February

Chapter 6

Discretisation

- Finite Elements
  - good for complex geometry
  - need good elliptic solver on unstructured grid
  - commercial code : POLYFLOW
- Spectral
  - very accurate
  - only for periodic geometry
    – wavy-wall tube, turbulent drag reduction
- Finite differences
  - simple, so good for understanding underlying difficulties
  - only for simple geometry (but mappable)

Finite Elements

- Divide domain into elements – triangles, quadrilaterals
- Represent unknowns by simple functions over elements

$$\mathbf{u}(\mathbf{x}) = \sum^{N} \mathbf{f}_i \phi_i(\mathbf{x})$$

E.G. for a triangle $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$,
$\phi_1(\mathbf{x}) = 1$ at vertex $\mathbf{x} = \mathbf{x}_1$ and vanishing at $\mathbf{x}_2$ and $\mathbf{x}_3$

$$\phi_1(\mathbf{x}) = \frac{(\mathbf{x} - \mathbf{x}_2) \times (\mathbf{x}_3 - \mathbf{x}_2) \cdot \hat{\mathbf{z}}}{(\mathbf{x}_1 - \mathbf{x}_2) \times (\mathbf{x}_3 - \mathbf{x}_2) \cdot \hat{\mathbf{z}}}$$

Finite Elements 2

- Substitute into momentum/mass/stress equation and project (Galerkin)

$$\int \left(\rho \frac{Du}{Dt} - \nabla \cdot \sigma \right) \cdot \phi_s(\mathbf{x})\, dV \;=\; 0, \qquad s \;=\; 1, 2, .., N$$

- Typical finite elements have less pressure modes than velocity, and sometimes more stress than velocity

Spectral

- Spectral representation (Fourier, or Chebyshev, or Stokes' eigensolutions)
$$f(x) = \sum^{N} f_n e^{inx}$$
- Possible problems with boundary conditions.
- Then differentiation
$$f'(x) = \sum^{N} f_n i n e^{inx} + O(e^{-N}) \qquad \text{good}$$
- but products
$$f(x)g(x) = \sum_{n}^{N} \sum_{k}^{N} f_k g_{n-k} e^{inx} \quad \text{bad}$$
- So use pseudo-spectral – evaluate products in real space and derivatives in Fourier space.

Spectral 2

- Galerkin or collocation to satisfy governing equations
- Fast Transforms useful
- Smooth OK, discontinuities bad (hidden at boundaries?)
- Aliasing – chop top $\frac{1}{3}$ of spectrum

Finite Differences

- Simple
- Needs coordinate grid
  - gives organised labelling
  - consider conformal map
- Differentiation – central 2^{nd} order
$$f'' \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$
- Conservative, e.g.
$$\nabla^4 \psi = \nabla \times \nabla \cdot (\nabla + \nabla^T) \nabla \times \psi \neq \nabla^2 \nabla^2 \psi$$

Fractional time-step

Pressure ensures incompressibility

Half step to u^* using no-slip BC

$$\frac{u^* - u^n}{\Delta t} = -(u\cdot\nabla u)^n + \nabla\cdot\sigma^n$$

Project to incompressible

$$u^{n+1} = u^* - \Delta t\nabla p^{n+1}, \qquad \text{so} \quad \nabla\cdot u^{n+1} = 0$$

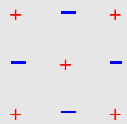
i.e. solve

$$\Delta t\nabla^2 p^{n+1} = \nabla\cdot u^*$$

Also pressure update $O(\Delta t^2)$

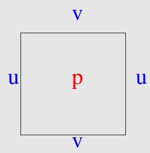
FD pressure problems

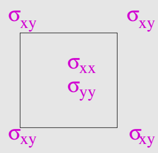
Spurious pressure modes



“$\nabla p = 0$”

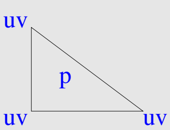
Avoided by staggered grid



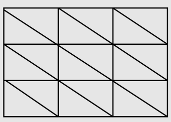


FE pressure problems

- Spurious pressure modes with “$\nabla p = 0$” – no staggered FE
- Locking



One Δ has $1p + 3u + 3v$



All grid has $18p + 4u + 4v$
if no-slip bc

Use ‘bubble elements’ with extra u, v at centre of triangles

Elliptic

Write EVSS = Elastic Viscous Split Stress

$$\sigma = -pI + 2\mu E + \sigma^{\text{elastic}},$$

where μ can be arbitrary and σ^{elastic} the remainder.

Then instantaneous Stokes flow driven by elastic stress

$$-\nabla p + \mu\nabla^2 u = -\nabla\cdot\sigma^{\text{elastic}}$$

Need fast elliptic solver

- conjugate gradients
- multigrid
- domain decomposition

Elliptic part 2

- Possible $\mu(x)$
- Possible anisotropic μ, e.g. FENE $AI + IA$
- Fast relaxed modes

$$\mu = \mu_0 + \sum_{\tau_i \ll \dot{\gamma}^{-1}} G_i \tau_i$$

Hyperbolic part

Stress equation is hyperbolic PDE

$$\frac{D\sigma}{Dt} = F(\sigma, \nabla u) \qquad \text{minor difficulty}$$

or streamwise integral equation (but DE better)

$$\sigma(t) = \int^t G(t-s) A^T A_{ts} Dt$$

Finite Differences

- second-order with 'flux-limiters', e.g. MINMOD
- use characteristics = streamlines

Hyperbolic part 2

Finite Elements

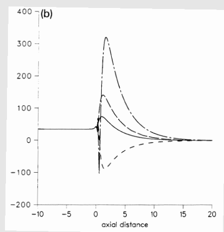
- PUPG – Streamline Upwinding Petrov Galerkin:

  $$\int (\text{stress equation}) \cdot (\phi + h\hat{u} \cdot \nabla\phi)\, dV = 0,$$

  but large numerical diffusion
- Lagrangian FE
  - exact $\int \nabla u Dt$
  - needs regridding
  - no fast elliptic solver

Hyperbolic part 3

Typical erroneous treatment of hyperbolic stress equation

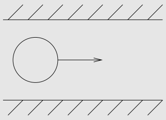


Continuous curve is correct solution.
Others have spurious oscillations.

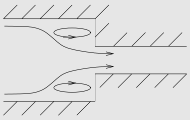
Bench marks

International campaign tackling bench-mark problems

1. Sphere in a tube, 2:1 diam
Dominated by shear

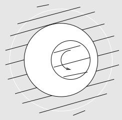


2. Contraction, 4:1
Difficult sharp corner

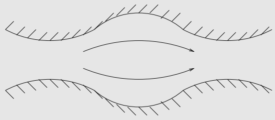


Bench marks 2

3. Journal bearing
Good for spectral



4. Wavy-wall pipe
Good for spectral



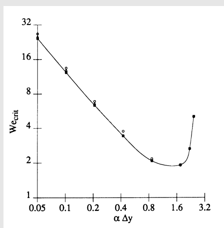
Eventually different algorithms produced the same results!

Numerical problems

- Convergence tests rarely done (well)
- New numerical instability
- Corner singularity → mess downstream
- Thin layers of high stress
- Limiting (maximum) value of De, e.g. sphere in a tube:
  - UCM $De_{\mathrm{max}} = 2.17$
  - O-B $De_{\mathrm{max}} = 1.28$ Fan (2003) JNNFM 110

Numerical problems 2

New numerical instability
Plotting σ_{xx}/σ_{xy} vs $\Delta y/\Delta x$

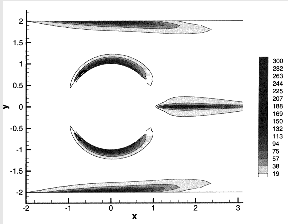


Need $\Delta y < \Delta x \dfrac{\sigma_{xy}}{\sigma_{xx}}$ to resolve direction of large N_1

Numerical problems 3

Thin layers of high stress

Flow past a sphere in a tube



Need to resolve

Other problems

- Need FENE modification of Oldroyd-B to avoid negative viscosities
- Smooth corners in contraction flow
- Contraction $\rightarrow$ Expansion, avoids long relaxation distance
- Micro-Macro Brownian fields, with same random Brownian forces in all spatial blocks, see later